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VALUED FIELDS – EXERCISE 12

To be submitted on Wednesday 2.2.2011 by 14:00 in the mailbox.

Definition.

(1) We say a valued field (K_2, ν_2) is an *extension* of (K_1, ν_1) when $O_1 = K_1 \cap O_2$. Denote this by $(K_1, \nu_1) \subseteq (K_2, \nu_2)$.
(2) Suppose (K_1, ν_1) is a valued field. We call an extension of fields $K_2 \supseteq K_1$ *finite-valued* if there are only finitely many valuations ν_2 such that $(K_2, \nu_2) \supseteq (K_1, \nu_1)$.

Question 1.

The aim of this question is to prove the following lemma:

- Suppose $K_1 \subseteq K_2$ is an algebraic extension. Suppose ν is a valuation of K_1 (with valuation ring O_1), and u, u' are two valuations of K_2 (with valuation rings O', O'') such that $(K_1, \nu) \subseteq (K_2, u)$ and $(K_1, \nu) \subseteq (K_2, u')$. Suppose $O' \subseteq O''$. Then $O' = O''$.

Use the following steps:
Let m', m'' be the maximal ideals of O', O'' resp. We know that $m'' \subseteq m'$ (why?). Let $k'' = O''/m''$, and let $o' = O'/m''$.

(1) Let $k = O_1/m$. Deduce that $k \subseteq o' \subseteq k''$ and that the extension k''/k is algebraic.
(2) Conclude that o' is a field and hence $m' = m''$ and finish. (Hint: see Question 3, clause (3) in Exercise 5).

Question 2.

(1) Show that if K_2/K_1 is finite-valued then the field extension K_2/K_1 is algebraic.
(2) Suppose K_2/K_1 is finite. Show that the map $\Delta \mapsto \Delta \cap \Gamma_1$ is an inclusion preserving bijection between the set of all convex subgroups of Γ_2 onto the set of all convex subgroups of Γ_1, and conclude that the rank of K_1 equals the rank of K_2.

Question 3.

Suppose ν is a non-trivial valuation on $\mathbb{R}$. Show that the residue field k is algebraically closed and that the valuation group Γ is a divisible group (i.e. if $\gamma \in \Gamma$ and $n \in \mathbb{N}$ then there exists $\gamma' \in \Gamma$ such that $n\gamma' = \gamma$).
Hint: use the same hint from Exercise 11, Question 3, and think.

Question 4.

(1) In class you have seen the following lemma: Suppose $O_1, \dots, O_n$ are valuation rings of a field K with $m_1, \dots, m_n$ maximal ideals. Let $R = \bigcap_{1 \leqslant i \leqslant n} O_i$ and $p_i = R \cap m_i$. Then for all $1 \leqslant i \leqslant n$, $O_i = R_{p_i}$. Give an easy proof of this lemma when there is some i such that $O_i \subseteq O_j$ for all j.

(2) Find 2 valuation rings of some field K whose intersection is not a valuation ring.