Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

VALUED FIELDS – EXERCISE 1

To be submitted on Wednesday 3.11.2010 by 14:00 in the mailbox.

Definition.

(1) A ring A is called Noetherian if every ideal I is finitely generated.
(2) Let A be a sub-ring of a field Ω. An element $\alpha \in \Omega$ is said to be integral over A if it satisfies a monic polynomial with coefficients from A: if there exists a relation of the form $\alpha^m + a_1\alpha^{m-1} + \cdots + a_m = 0$ where $a_i \in A$.
(3) A ring $A \subseteq B \subseteq \Omega$ is an integral extension of A if all its elements are integral over A.
(4) The ring A is said to be *integrally closed in* Ω if every element of Ω which is integral over A is already in A.
(5) A domain A is said to be *integrally closed* if it is integrally closed in its field of fractions.
(6) A domain A is called a Dedekind domain if it satisfies the following conditions:
   (a) It is integrally closed.
   (b) Every nonzero prime ideal $\mathfrak{p}$ in A is maximal.
   (c) It is a Noetherian ring.
(7) A *fractional ideal* in a domain A is a finitely generated A sub-module of $\mathrm{quot}(A)$ (the field of fractions).
(8) If $x \in \mathrm{quot}(A)$, then (x) is the fractional ideal generated by x, namely it is $xA = \{xa \mid a \in A\}$.
(9) If I, J are fractional ideals, then their product is the module containing all finite sums of the form $\sum a_i b_i$ where $a_i \in I, b_i \in J$. (check that it is also a fractional ideal).

Comments: For us, a ring is always commutative, and an ideal is never the whole ring.

Question 1.
Let A be a ring. Prove that the following are equivalent:

(1) A is Noetherian.
(2) For every nonempty set of ideals of A, P, there is a maximal element.
(3) There is no infinite increasing chain $I_0 \subsetneq I_1 \subsetneq I_2 \ldots$ of ideals.

Question 2.
Let A be a sub-ring of a field Ω, and $\alpha \in \Omega$. Prove that the following are equivalent:

(1) α is integral over A.
(2) The ring generated by α over A, denoted $A[\alpha]$, is finitely generated as an A-module.
(3) There exists a finitely generated nonzero A-module $M \subseteq \Omega$ such that $\alpha M \subseteq M$.

Hint: for (3) implies (1): Suppose M is generated by $\{\omega_1, \dots, \omega_n\}$, and that $\alpha\omega_i = \sum_{j=1}^{n} a_{i,j}\omega_j$. Consider the matrix $B = \langle a_{i,j} \rangle$, show that α solves the monic polynomial $m(x) = \det(B - xI)$.

Question 3.

(1) Conclude from Question 2 that if $A \subseteq \Omega$, then the set $\{b \in \Omega \,|b \text{ is integral over } A\}$ is a ring.
Hint: Note that if $A[\alpha]$ and $A[\beta]$ are finitely generated, then so is $A[\alpha, \beta]$.
(2) Conclude from Question 2 that if A is a PID (principle ideal domain) then A is integrally closed.
Hint: Suppose $\alpha = c/d \in \text{quot}(A)$ is integral over A, and let M be from (3). Show that $M = (a/b)A$ for some $a, b \in A$.
(3) Show directly from the definition that if A is a UFD (unique factorization domain) then it is integrally closed.
Hint: Suppose $\alpha = c/d \in \text{quot}(A)$ is integral over A, and c/d is reduced, and $m(\alpha) = 0$ where m is monic of degree n. Multiply both sides of the equation by d^{n-1}.

Question 4.

Suppose A is a Dedekind Domain. Let $K = \text{quot}(A)$.

(1) Prove that every nonzero ideal I in A contains a product of nonzero prime ideals.
Hint: Suppose not. Let I be an ideal, maximal with properties: $I \neq 0$ and I does not contain a product of nonzero primes (why does it exist?). We may assume I is not prime, so there are $a, b \in A$ such that $a \cdot b \in I$ but $a, b \notin I$. Then $I + (a), I + (b)$ strictly contain I, so they contain a product of nonzero primes. Now look at their product.
For a fractional ideal $0 \neq I$, define $I' = \{x \in K \,|xI \subseteq A\}$.
(2) Show that I' is also a fractional ideal of A.
Hint: for showing that I' is finitely generated: if $0 \neq c \in I$ then $I' \subseteq (1/c)A$, so as A is Noetherian, I' is finitely generated.
(3) Suppose P is a nonzero prime. Show that $P' \nsubseteq A$.
Hint: We may assume P is not zero. Suppose $0 \neq a \in P$. By (1), there are nonzero primes, $P_1, \dots, P_r$ with r minimal such that $(a) \supseteq P_1 \dots P_r$. It follows that $P \supseteq P_i$ for some i, so in fact $P = P_i$. Assume $i = 1$. By minimality, there is some $b \notin (a)$ but $b \in P_2 \dots P_r$. But $(a) \supseteq P(b)$, so $b/a \in P' \backslash A$.
(4) Suppose P is a nonzero prime. Show that $P'P = A$.
Hint: it is enough to show that $P'P \supseteq A$. Suppose not, then show that $P'P = P$. Then by Question 2, show that every element of P' is integral over A, and thus $P' \subseteq A$.