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VALUED FIELDS – EXERCISE 3

To be submitted on Wednesday 24.11.2010 by 14:00 in the mailbox.

Definition.

(1) A domain R is called a valuation ring if for any $x \in \text{quot}(R)$, either $x \in R$ or $1/x \in R$.
(2) A ring R is called local if it has a unique maximal ideal.

Question 1.
Prove directly from the definitions that every valuation ring is local.
Possible hint: It is enough to prove that if I is an ideal, x is not invertible in R, then $I + (x) \neq R$, hence it is enough to show that if x is not invertible then $1 + x$ is invertible.

The following definition is an abstraction of the notion of algebraic dependence, which we will lead us in Question 3 to the notion of transcendental degree of a ring.

Definition.

(1) A *pregeometry* consists of a set X, and a function cl (called closure) $\text{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ that satisfies the following conditions for all $a, b \in X$ and all $Y, Z \subseteq X$:
  (a) $Y \subseteq \text{cl}(Y)$
  (b) If $Y \subseteq Z$, then $\text{cl}(Y) \subseteq \text{cl}(Z)$.
  (c) $\text{cl}(\text{cl}(Y)) = \text{cl}(Y)$.
  (d) (finite character) If $a \in \text{cl}(Y)$, then there is a finite subset $Y' \subseteq Y$ such that $a \in \text{cl}(Y')$.
  (e) (exchange principle) If $a \in \text{cl}(Y \cup \{b\}) \backslash \text{cl}(Y)$, then $b \in \text{cl}(Y \cup \{a\})$.

Question 2.
Suppose cl is a pregeometry on a set X. Say a subset $B \subseteq X$ is independent if for each $b \in B$, $b \notin \text{cl}(B \backslash \{b\})$. B is said to be a basis of X if it is independent and $\text{cl}(B) = X$.

(1) Suppose cl is a pregeometry on a set X, and $Y \subseteq X$. Define $\text{cl}_Y(Z) = \text{cl}(Z) \cap Y$. Show that cl_Y is pregeometry on Y.
(2) Show that if $B = \{a_i \,|i = 1, \dots, n\}$ is such that $a_i \notin \text{cl}(\{a_1, \dots, a_{i-1}\})$ for every i, then B is an independent set.
(3) Show that for every pregeometry X, a basis B always exists and in fact any independent set can be extended into a basis.
Hint: take as basis a maximal independent set.
(4) Show that if B' and B are basis of X and $|B| < \infty$, then $|B| = |B'|$.
(5) Bonus: Show (4) without any assumption on $|B|$.

Definition. For a pregeometry cl on a set X, and for $Y \subseteq X$, define $\dim_{\text{cl}}(Y)$ to be the cardinality of any basis of Y in the pregeometry cl_Y.

Question 3.

(1) Let k be a field, and $K \supseteq k$. For a set $X \subseteq K$, define
$$\mathrm{cl}_k(X) = \{y \in K \,|\, y \text{ is algebraic over } k(X)\}.$$
Prove that cl is a pregeometry on subsets of K.
Hint: note that $\{a_1, \ldots, a_n\} \subseteq K$ is independent iff there is no polynomial $p(X_1, \ldots, X_n)$ over k such that $p(a_1, \ldots, a_n) = 0$.

(2) Suppose now that $K \supseteq R \supseteq k$ is a ring, such that $K = \mathrm{quot}(R)$. Show that if B is a basis for R in $(\mathrm{cl}_k)_R$ (see Question 1, clause (1)) , then B is also a basis for K in cl_k.
Namely, you should show that every element of K is algebraic over $k(B)$.

Definition. For a domain R containing a field k, let $\mathrm{tr.deg}_k(R) = \dim_{\mathrm{cl}_k}(R)$ (the transcendental degree of R over k).

(3) Show that if R is a domain generated by $a_1, \ldots, a_m$ as a ring over k then $\mathrm{tr.deg}_k(R) \leqslant m$.

Question 4.

Suppose R and R' are two integral domains containing a field k.

(1) Show that if $f : R \to R'$ is an isomorphism over k (i.e. $f(x) = x$ for all $x \in k$). Then $\mathrm{tr.deg}(R) = \mathrm{tr.deg}(R')$.

(2) Show that if $f : R \to R'$ is a surjective homomorphism over k, then $\mathrm{tr.deg}(R') \leqslant \mathrm{tr.deg}(R)$.
You may assume that both have finite tr. degree.

(3) Show that under the assumptions of (2) above, if $\mathrm{tr.deg}(R') = \mathrm{tr.deg}(R) < \infty$ then f is in fact an isomorphism.
Hint: Let $B' = \{a'_1, \ldots, a'_n\}$ be a basis of R', and let $B = f^{-1}(B) = \{a_1, \ldots, a_n\}$. Let $u \neq 0$ be in R, and let $p(X_1, \ldots, X_n, Y)$ be a polynomial over k such that $p(a_1, \ldots, a_n, u) = 0$ of minimal degree. So $p(X_1, \ldots, X_n, 0) \neq 0$. If $f(u) = 0$ then $p(a'_1, \ldots, a'_n, 0) = 0$ – a contradiction.

(4) Bonus: show that (3) is not true if we allow $\mathrm{tr.deg}(R) = \mathrm{tr.deg}(R')$ to be infinite.