Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

VALUED FIELDS – EXERCISE 13

To be submitted on Wednesday 9.2.2011 by 14:00 in the mailbox.

Definition.

(1) Let K be a field. A polynomial $f \in K[X]$ is said to *split* over K if f can be written as a product of linear polynomials, in other words, if $f = a\prod(X - b_i)$ for some $a, b_i \in K$.
(2) We say a field extension $K_2 \supseteq K_1$ is *normal* if it is algebraic and every irreducible polynomial over K_1 that has a root in K_2 splits over K_2.
(3) Suppose $L \supseteq K$ is a field extension. We denote by $\mathrm{Aut}(L/K)$ the group of automorphisms of L that fix K.

Question 1.

Suppose N/K is a normal extension, and let F_1, F_2 be two subfields of N containing K. Let $\psi : F_1 \to F_2$ be an isomorphism of fields that fixes K.

(1) Suppose $\beta \in N$. Prove that there are extensions $F_1' \supseteq F_1$ and $F_2' \supseteq F_2$ (both contained in N) and an isomorphism extending ψ, $\psi' : F_1' \to F_2'$ such that $\beta \in F_1'$.
Hint: let f be the minimal polynomial of β over F_1, show that $\psi(f)$ is irreducible over F_2, and that $\psi(f)$ divides the minimal polynomial of β over K.
(2) Suppose $\beta \in N$. Prove that there are extensions $F_1' \supseteq F_1$ and $F_2' \supseteq F_2$ (both contained in N) and an isomorphism extending ψ, $\psi' : F_1' \to F_2'$ such that $\beta \in F_2'$.
(3) Use Zorn's lemma to show that there is an extension of ψ to an automorphism of N fixing K.
Hint: define the set
$\{(F_1', F_2', \psi') \mid F_1 \subseteq F_1' \subseteq N, F_2 \subseteq F_2' \subseteq N, \psi \subseteq \psi' : F_1' \xrightarrow{\sim} F_2'\}$.

Question 2.

Let N/K be a normal extension. Let K^{sep} be the separable closure of K inside N.

(1) Show that K^{sep}/K is normal.
(2) Let $G = \mathrm{Aut}(N/K)$, and $H = \mathrm{Aut}(K^{\mathrm{sep}}/K)$. Show that the restriction map

$$\psi \mapsto \psi|_{K^{\mathrm{sep}}}$$

is a well defined isomorphism from G to H.
Hint: use Question 1 for surjectivity and (1) for well-definiteness. For injectitivity, note that if $x \in N$, then if the char. of K is p, then for some e, $x^{p^e} \in K^{\mathrm{sep}}$.

Question 3.

Under the assumptions of Question 2, suppose that (K, O) is a valued field.

(1) Suppose that
$\star(K^{\mathrm{sep}})$ For every 2 valuation rings $O_1, O_2 \subseteq K^{\mathrm{sep}}$ such that $(K, O) \subseteq (K^{\mathrm{sep}}, O_1)$ and $(K, O) \subseteq (K^{\mathrm{sep}}, O_2)$, there is some $\sigma \in \mathrm{Aut}(K^{\mathrm{sep}}/K)$ such that $\sigma(O_1) = O_2$.

Show that $\star(N)$ is also true (i.e. after replacing K^{sep} by N).

(2) Deduce from a theorem proved in class that $\star(N)$ is true for all finite normal extensions N/K.

Question 4.

Let $L \supseteq K$ be an algebraic extension. Let O be a valuation ring of K. Let R be the integral closure of O in L and let O' be an extension of O to L. Let M be the maximal ideal of O' and let $\mathfrak{m} = R \cap M$. Show that $R_{\mathfrak{m}} = O'$.

Hints:

(1) First show $R_{\mathfrak{m}} \subseteq O'$.

(2) For the other direction, suppose $\alpha \in O'$. By letting $L' = K(\alpha)$, show that you can reduce to the situation where L/K is finite.

(3) Show that in general, if $L \subseteq F$ is a field extension, O_L a valuation ring on L (with maximal ideal $\mathfrak{m}_L$) and $O_F \supseteq O_L$ is a valuation ring on F (with maximal ideal $\mathfrak{m}_F$) then $(L, O_L) \subseteq (F, O_F)$ (i.e. $O_F \cap L = O_L$) iff $\mathfrak{m}_F \cap O_L = \mathfrak{m}_L$.

(4) Deduce that in the situation of (1), the integral closure of O_L in F is the intersection of all valuation rings of O_F of F such that $(L, O_L) \subseteq (F, O_F)$. (Hint: find a theorem about places that we had earlier in the semester).

(5) Now use a lemma given in class to conclude.