

POSITIVE POLYNOMIALS LECTURE NOTES

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1. ARCHIMEDEAN MODULES

Let A be a commutative ring, $\mathbb{Q} \subseteq A$, T a preprime.

Definition 1.1. Let M a T -module. M is **archimedean** if:

$$\forall a \in A, \exists N \geq 1, N \in \mathbb{Z}_+ \text{ s.t. } N + a, N - a \in M .$$

Proposition 1.2. Let T be a generating preprime, M a maximal proper T -module. Assume that M is archimedean. Then \exists a uniquely determined $\alpha \in \text{Hom}(A, \mathbb{R})$ s.t. $M = \alpha^{-1}(\mathbb{R}_+) = P_\alpha$.
(In particular, M is an ordering, not just a semi-ordering.)

Proof. Let $a \in A$, define:

$\text{cut}(a) = \{r \in \mathbb{Q} \mid r - a \in M\}$, this is an **upper cut** in \mathbb{Q} (i.e. final segment of \mathbb{Q}) .

Claim 1: $\text{cut}(a) \neq \emptyset$ and $\mathbb{Q} \setminus (\text{cut}(a)) := L(a) \neq \emptyset$, where $L(a)$ is a **lower cut** in \mathbb{Q} .

Proof of claim 1. Since M is archimedean $\exists n \geq 1$ s.t. $n - a \in M$, so $\text{cut}(a) \neq \emptyset$.

Also $\exists m \geq 1$ s.t. $(m + a) \in M$.

If $-(m + 1) - a \in M$, then adding we get $-1 \in M$, a contradiction (since M is proper). So we have $-(m + 1) - a \notin M$, which $\Rightarrow -(m + 1) \in L(a)$.

□(claim 1)

Now define a map $\alpha : A \rightarrow \mathbb{R}$ by

$$\alpha(a) := \inf (\text{cut}(a))$$

α is well-defined.

Claim 2: $\alpha(1) = 1$, $\alpha(M) \subseteq \mathbb{R}_+$; $\alpha(a \pm b) = \alpha(a) \pm \alpha(b) \forall a, b \in A$ and $\alpha(tb) = \alpha(t) \alpha(b) \forall t \in T, b \in A$.

This is left as an exercise.

Claim 3: $\alpha(ab) = \alpha(a) \alpha(b) \forall a, b \in A$

Proof of claim 3. T generating $\Rightarrow a = t_1 - t_2, t_1, t_2 \in T$

so, $\alpha(ab) = \alpha(t_1b - t_2b) = \alpha(t_1a) - \alpha(t_2b)$

$$= \alpha(t_1)\alpha(b) - \alpha(t_2)\alpha(b) \text{ [by claim 2]}$$

$$= (\alpha(t_1) - \alpha(t_2))\alpha(b) = \alpha(t_1 - t_2)\alpha(b) = \alpha(a)\alpha(b) .$$

□(claim 3)

Claim 4: $\alpha^{-1}(\mathbb{R}_+) = M$

Proof of claim 4. By Claim 2, $M \subseteq \alpha^{-1}(\mathbb{R}_+)$

so, by maximality of M and since $P_\alpha = \alpha^{-1}(\mathbb{R}_+)$ is an ordering it follows that

$$M = \alpha^{-1}(\mathbb{R}_+) .$$

□

Corollary 1.3. Let A be a commutative ring with 1, T an archimedean preprime, M a T -module, $-1 \notin M$ (i.e. M proper T -module). Then $\chi_M \neq \emptyset$.

Proof. Since T is archimedean, T is generating (because $a = (n + a) - n$, for $a \in A$) and M is a proper archimedean module (archimedean module because for an archimedean preprime T , every T -module is also archimedean). By Zorn's lemma extend M to a maximal proper archimedean module Q . Apply Proposition 1.2 to Q to get $\alpha \in \text{Hom}(A, \mathbb{R})$ such that $Q = \alpha^{-1}(\mathbb{R}_+)$. This implies $M \subseteq \alpha^{-1}(\mathbb{R}_+)$. So, $\alpha \in \chi_M$, which implies $\Rightarrow \chi_M \neq \emptyset$. □

2. REPRESENTATION THEOREM (STONE-KRIVINE, KADISON-DUBOIS)

The following corollary (to Proposition 1.2 and Corollary 1.3) answers the question raised in 2.4 of lecture 12:

Corollary 2.1. (Stone-Krivine, Kadison-Dubois) Let A be a commutative ring, T an archimedean preprime in A , M a proper T -module. Let $a \in A$ and

$$\hat{a} : \chi \rightarrow \mathbb{R} \quad \text{defined by}$$

$$\hat{a}(\alpha) := \alpha(a)$$

If $\hat{a} > 0$ on χ_M , then $a \in M$.

Proof. Assume $\hat{a} > 0$ on χ_M , i.e. $\hat{a}(\alpha) > 0 \forall \alpha \in \chi_M$.

To show: $a \in M$

- Consider $M_1 := M - aT$

Since $\alpha(a) > 0 \forall \alpha \in \chi_M$, we have $\chi_{M_1} = \emptyset$ [because if $\alpha \in \chi_{M_1}$, then $\alpha(M_1) \subseteq \mathbb{R}_+$. So, $\alpha(-a) = -\alpha(a) \geq 0$. So, $\alpha(a) \leq 0$, but $\alpha \in \chi_M$ so $\alpha(a) > 0$, a contradiction].

So (since M_1 is an archimedean T -module), we can apply Corollary 1.3 to M_1 to deduce that $-1 \in M_1$.

Write $-1 = s - at$, $s \in M, t \in T$

$$\Rightarrow at - 1 = s \in M \tag{*}$$

- Consider $\Sigma := \{r \in \mathbb{Q} \mid r + a \in M\}$

We **claim** that: $\exists \rho \in \Sigma; \rho < 0$

Once the claim is established we are done (with the proof of corollary) because

$$a = \underbrace{(a + \rho)}_{\in M} + \underbrace{(-\rho)}_{\in M} \in M.$$

Proof of the claim: First observe that $\Sigma \neq \emptyset$ (since $\exists n \geq 1$ s.t. $n+a \in T \subseteq M$, so $n \in \Sigma$).

Now fix $r \in \Sigma$, $r \geq 0$ and fix an integer $k \geq 1$ s.t. $(k-t) \in T$

$$\text{Write: } kr - 1 + ka = \underbrace{(k-t)}_{\in T} \underbrace{(r+a)}_{\in M} + \underbrace{(at-1)}_{\in M} + \underbrace{rt}_{\in M} \in M$$

Multiplying by $\frac{1}{k}$, we get

$$\left(r - \frac{1}{k}\right) + a \in M, \text{ i.e. } \left(r - \frac{1}{k}\right) \in \Sigma$$

Repeating we eventually find $\rho \in \Sigma, \rho < 0$. □

Note 2.2. For a quadratic module $M \subseteq \mathbb{R}[\underline{X}]$, set

$$K_M := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \forall g \in M\}.$$

Note that if $M = M_S$ with $S = \{g_1, \dots, g_s\}$, then $K_S = K_M$.

We have the following corollaries to Corollary 2.1. (Stone-Krivine, Kadison-Dubois):

Corollary 2.3. (Putinar's Archimedean Positivstellensatz) Let $M \subseteq \mathbb{R}[\underline{X}]$ be an archimedean quadratic module. Then for each $f \in \mathbb{R}[\underline{X}]$:

$$f > 0 \text{ on } K_M \Rightarrow f \in M .$$

Corollary 2.4. Let $A = \mathbb{R}[\underline{X}]$ and $S = \{g_1, \dots, g_s\}$. Assume that the finitely generated preordering T_S is archimedean. Then for all $f \in A$:

$$f > 0 \text{ on } K_S \Rightarrow f \in T_S.$$

Remark 2.5.

1. To apply the corollary we need a criterion to determine when a preordering (quadratic module) is archimedean.
2. T_S is archimedean \Rightarrow for $f = \sum X_i^2 : \exists N$ s.t. $N - f = N - \sum X_i^2 \in T_S$
 $\Rightarrow N - \sum X_i^2 \geq 0$ on K_S .
 $\Rightarrow K_S$ is bounded. Also K_S is closed.
 So T_S is archimedean implies K_S is compact.