REAL ALGEBRAIC GEOMETRY LECTURE NOTES (09: 17/11/09 - BEARBEITET 26/11/18)

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Contents

1. Basic version of Tarski-Seidenberg

Basic version: Let $(R, \leqslant)$ be a real closed field. We are interested in a system of equations and inequalities (*Gleichungen und Ungleichungen*) for $\underline{X} = (X_1, \dots, X_n)$ of the form

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

where $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ and $f_i(\underline{X}) \in \mathbb{Q}[\underline{X}]$ or $f_i(\underline{X}) \in R[\underline{X}]$. We say that $S(\underline{X})$ is a system of polynomial equalities and inequalities with coefficients in $\mathbb{Q}$ (or with coefficients in R) in n variables.

Theorem 1.1. *(Tarski-Seidenberg Theorem: Basic Version) Let $S(\underline{T}; \underline{X})$ be a system with coefficients in $\mathbb{Q}$ in $m+n$ variables, with $\underline{T} = (T_1, \dots, T_m)$ and $\underline{X} = (X_1, \dots, X_n)$. Then there exist $S_1(\underline{T}), \dots, S_l(\underline{T})$ systems in m variables and coefficients in $\mathbb{Q}$ such that:*

for every real closed field R and every $\underline{t} = (t_1, \dots, t_m) \in R^m$ the system $S(\underline{t}; \underline{X})$ of polynomial equalities and inequalities in n variables and coefficients in R obtained by substituting T_i with t_i in $S(\underline{T}, \underline{X})$ for every $i = 1, \dots, m$, has a solution $\underline{x} = (x_1, \dots, x_n) \in R^n$ if and only if $\underline{t} = (t_1, \dots, t_m) \in R^m$ is a solution in R for one of the systems $S_1(\underline{T}), \dots, S_l(\underline{T})$.

Example 1.2. Let $m = 3$ and $n = 1$, so $\underline{T} = (T_1, T_2, T_3)$ and $\underline{X} = X$, and

$$S(\underline{T}, \underline{X}) := \begin{cases} T_1 X^2 + T_2 X + T_3 = 0 \end{cases}$$

Let R be a real closed field and $(t_1,t_2,t_3)\in R^3$. Then $S(\underline{t};X)$ has a solution in R if and only if

$$\underset{S_1(T_1,T_2,T_3)}{(t_1\neq 0 \ \wedge\ t_2^2-4t_1t_3\geqslant 0)} \quad\vee\quad \underset{S_2(T_1,T_2,T_3)}{(t_1=0\ \wedge\ t_2\neq 0)} \quad\vee\quad \underset{S_3(T_1,T_2,T_3)}{(t_1=t_2=t_3=0)}$$

Concise version:

$$\forall\underline{T}\ [\ (\exists\,\underline{X}:S(\underline{T};\underline{X}))\ \Leftrightarrow\ (\bigvee_{i=1}^{l}S_i(\underline{T}))\ ].$$

Remark 1.3. The proof is by induction on n.

The case $n=1$ is the heart of the proof and we will show it later.

For now, let us just convince ourselves that the induction step is straightforward.

Assume $n>1$, so

$$S(\underline{T};X_1,\dots,X_n)=S(\underline{T},X_1,\dots,X_{n-1};X_n).$$

By case $n=1$ we have finitely many systems $S_1(\underline{T},X_1,\dots,X_{n-1}),\dots,S_l(\underline{T},X_1,\dots,X_{n-1})$ such that

for any real closed field R and any $(t_1,\dots,t_m,x_1,\dots,x_{n-1})\in R^{m+n-1}$ we have

$$\exists\,X_n:\ S(t_1,\dots,t_m,x_1,\dots,x_{n-1};X_n)\ \longleftrightarrow\ \bigvee_{i=1}^{l}S_i(t_1,\dots,t_m,x_1,\dots,x_{n-1}).$$

By induction hypothesis on $n-1$:

for every fixed i, $1\leqslant i\leqslant l$, $\exists$ systems $S_{ij}(\underline{T})$, $j=1,\dots,l_i$ such that: for each real closed field R and each $\underline{t}\in R^m$ the system

$$S_i(\underline{t};X_1,\dots,X_{n-1})$$

has a solution $(x_1,\dots,x_{n-1})\in R^{n-1}$ if and only if $\underline{t}$ is a solution for one of the systems $S_{ij}(\underline{T})$; $j=1,\dots,l_i$.

Therefore for any real closed field R and any $\underline{t}\in R^m$

$$S(\underline{t};X_1,\dots,X_n)\text{ has a solution }\underline{x}\in R^n\text{ if and only if}$$

$$\underline{t}\text{ is a solution to one of the systems }\{S_{ij}(\underline{T});\ i=1,\dots,l,\ j=1,\dots,l_i\}$$

2. Tarski Transfer Principle I

Theorem 2.1. *Let $S(\underline{T},\underline{X})$ be a system with coefficients in $\mathbb{Q}$ in $m+n$ variables. Let $(K,\leqslant)$ be an ordered field. Let R_1,R_2 be two real closed extensions of $(K,\leqslant)$. Then for every $\underline{t}\in K^m$, the system $S(\underline{t},\underline{X})$ has a solution $\underline{x}\in R_1^n$ if and only if it has a solution $\underline{x}\in R_2^n$.*

Proof. Let $\underline{t} \in K^m \subseteq R_1^m \cap R_2^m$. Then there are systems $S_i(\underline{T})$ $(i = 1, \dots, l)$ with coefficients in $\mathbb{Q}$ and variables $T_1, \dots, T_m$ such that

$$\exists\, \underline{x} \in R_1 : S(\underline{t}, \underline{x}) \;\longleftrightarrow\; \underline{t} \;\text{ satisfies } \bigvee_{i=1}^{l} S_i(\underline{T}) \;\longleftrightarrow\; \exists\, \underline{x} \in R_2 : S(\underline{t}, \underline{x}).$$

□

3. Tarski Transfer Principle II

Theorem 3.1. *Let $(K, \leqslant)$ be an ordered field, R_1, R_2 two real closed extensions of $(K, \leqslant)$. Then a system of polynomial equations and inequalities of the form*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

where $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ *and* $f_i(\underline{X}) \in K[X_1, \dots, X_n]$,

has a solution $\underline{x} \in R_1^n \iff$ *it has a solution* $\underline{x} \in R_2^n$.

Proof. Let $t_1, \dots, t_m$ be the coefficients of the polynomials $f_1, \dots, f_k$, listed in some fixed order. Replacing the coefficients $t_1, \dots, t_m$ by variables $T_1, \dots, T_m$ yields a system $\sigma(\underline{T}, \underline{X})$ in $m + n$ variables with coefficients in $\mathbb{Q}$ (in fact in $\mathbb{Z}$) for which

$$\sigma(t_1, \dots, t_m, \underline{X}) = S(\underline{X}).$$

Now we can apply Tarski Transfer I. □

4. Tarski Transfer Principle III

Theorem 4.1. *Suppose that $R \subseteq R_1$ are real closed fields. Then a system of polynomial equations and inequalities with coefficients in R*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

where $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ *and* $f_i(\underline{X}) \in R[X_1, \dots, X_n]$

has a solution $\underline{x} \in R_1^n \iff$ *it has a solution* $\underline{x} \in R^n$.

Proof. Apply Tarski Transfer II with $K = R_2 = R$. □

5. Tarski Transfer Principle IV

Theorem 5.1. *Let R be a real closed field and $(F, \leqslant)$ an ordered field extension of R. Then a system of polynomial equations and inequalities of the form*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

where $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ *and* $f_i(\underline{X}) \in R[X_1, \dots, X_n]$

$$\text{has a solution } \underline{x} \in F^n \iff \text{it has a solution } \underline{x} \in R^n.$$

Proof. Let R_1 be the real closure of the ordered field $(F, \leqslant)$ and apply Tarski Transfer III. □

6. Lang's Homomorphism Theorem

Corollary 6.1. *Suppose R and R_1 are real closed fields, $R \subseteq R_1$. Then a system of polynomial equations of the form*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) = 0 \\ \quad \vdots \\ f_k(\underline{X}) = 0 \end{cases} \qquad f_i(\underline{x}) \in R[X_1, \dots, X_n]$$

has a solution $\underline{x} \in R_1^n$ if and only if it has a solution $\underline{x} \in R^n$.

Proof. Apply Tarski Transfer III. □

The previous Corollary is equivalent to the following:

Theorem 6.2. *(Homomorphism Theorem I). Let R and R_1 be real closed fields, $R \subseteq R_1$. For any ideal $I \subseteq R[\underline{X}]$, if there exists an R-algebra homomorphism*

$$\varphi\colon R[\underline{X}]/I \longrightarrow R_1$$

then there exists an R-algebra homomorphism

$$\psi\colon R[\underline{X}]/I \longrightarrow R.$$

Proof. By Hilbert's Basis Theorem, I is finitely generated, say $I = \langle f_1, \dots, f_k \rangle$, with $f_1, \dots, f_k \in R[\underline{X}]$. Consider the system

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) = 0 \\ \quad \vdots \\ f_k(\underline{X}) = 0 \end{cases}$$

Claim. There is a bijection

$$\{\underline{x} \in R_1^n \text{ solution to } S(\underline{X})\} \longleftrightarrow \{\varphi\colon R[\underline{X}]/I \to R_1 \;\; R\text{-algebra homomorphism}\}$$

Proof of the claim:

Let $\underline{x} \in R_1^n$ be a solution to $S(\underline{X})$; then the evaluation homomorphism

$$\begin{array}{cccc} \varphi\colon & R[\underline{X}]/I & \longrightarrow & R_1 \\ & f+I & \mapsto & f(\underline{x}) \end{array}$$

is well-defined and is an R-algebra homomorphism.

Conversely: assume that

$$\varphi\colon R[\underline{X}]/I \longrightarrow R_1$$

is an R-algebra homomorphism. Then for $\underline{e} = (e_1, \ldots, e_n)$ and $f = \sum \underline{a}_e \underline{X}^e = \sum a_{e_1 \ldots e_n} X_1^{e_1} \ldots X_n^{e_n} \in R[\underline{X}]$,

$$\varphi(f+I) = \sum \underline{a}_e \varphi(X_1+I)^{e_1} \cdots \varphi(X_n+I)^{e_n} = f(\varphi(X_1+I), \ldots, \varphi(X_n+I)).$$

In other words set $(x_1, \ldots, x_n) \in R_1^n$ to be defined by $x_1 := \varphi(X_1 + I), \ldots, x_n := \varphi(X_n + I)$, then $(x_1, \ldots, x_n)$ is a solution to $S(\underline{X})$ and the R-algebra homomorphism φ is indeed given by point evaluation at $\underline{x} = (x_1, \ldots, x_n) \in R_1^n$.

Now apply Corollary 6.1. $\square$