Categoricity of AEC, analytic geometry and topological invariants

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Model theory and Geometry

- First-order-language categoricity $\rightarrow$ algebraic geometry
- ? $\rightarrow$ analytic (complex) theory
- Shelah's categoricity and stability theory $\rightarrow$?
- categoricity in $L_{\omega_1,\omega}$-languages[1] $\rightarrow$ analytic theory

[1] $L_{\omega_1,\omega}$: allows countable conjunctions and disjunctions. E.g.

$$\exists a \in G\, \forall x \in G\, (x = a \vee x = a^2 \vee x = a^3 \vee \ldots)$$

Case studies

A basic example:

$$\mathbb{C}_{\mathsf{exp}} := (\mathbb{C}; +, \cdot, \mathsf{exp}).$$

In order to understand the structure such as $\mathbb{C}_{\mathsf{exp}}$ one needs first to work out a complete (i.e. categorical) $L_{\omega_1,\omega}$-axiomatisation of the respective **cover structure.**

The cover structure

$$\begin{tikzcd} \mathbb{U} \arrow[d, "f"] \\ \mathbb{X} \end{tikzcd}$$

algebraic subvarieties of $\mathbb{X}^n$ (all n) definable

The cover structure

$$\begin{tikzcd}
 & \Gamma \circlearrowright \mathbb{U} \arrow[dl, "f_m"'] \arrow[ddd, "f"] \arrow[ddr, "f_n"] & \\
\Gamma/\Gamma(m) \circlearrowright \mathbb{X}_m \arrow[drr, "\mathrm{pr}_{m,n}"] \arrow[ddr, "\mathrm{pr}_{m,1}"'] & & \\
 & & \mathbb{X}_n \circlearrowleft \Gamma/\Gamma(n) \arrow[dl, "\mathrm{pr}_{n,1}"] \\
 & \mathbb{X} &
\end{tikzcd}$$

e.g. $\mathbb{U} = \mathbb{C},\ f_n(z) = \exp(\frac{z}{n})$

$\mathbb{X}_n = \mathbb{C}^*,\ \Gamma/\Gamma(n) \cong \mathbb{C}_n$

The cover structure

Claim. This structure holds a lot of information about the metric topology on $\mathbb{X}(\mathbb{C})$ and at the same time is stable (that is essentially algebro-geometric).

Note that without the cover $\mathbb{U}$ we only know about étale topology of $\mathbb{X}$, that is $\mathbb{X}$ up to abstract automorphisms of $\mathbb{C}$.

Some history

Theorem: *The* **natural** $L_{\omega_1,\omega}$*-theory of covers in basic cases (* $\exp$, $\mathfrak{p}$, j, *...) is categorical in uncountable cardinals.*

Proofs require Shelah's theory of AEC (with some important extensions) plus some strong **arithmetic theorems**:

1. (Proved) versions of the Mumford-Tate conjecture
2. Extensions of Galois and Kummer theories

(Z., Kirby, Gavrilovich, Bays, Harris, Daw, Hart, Haykazian, Hyttinen, Eterovich,...)

In the above, the required arithmetic facts are **sufficient and necessary***:*

Assuming that the **natural** *theory is categorical, the arithmetic facts follow.*

What further to expect?

¿For any smooth complex algebriac variety $\mathbb{X}$ with **non-trivial cover** *there is an $L_{\omega_1,\omega}$-axiomatisation $\Sigma(\mathbb{X})$ of the cover of $\mathbb{X}$ which is categorical in all uncountable cardinals*?

A recent result

Theorem (2022) *For any smooth complex projective curve $\mathbb{X}$ of genus ≥ 1 there is an $L_{\omega_1,\omega}(Q)$-axiomatisation $\Sigma^?(\mathbb{X})$ of the universal cover of $\mathbb{X}$ which is categorical in all uncountable cardinals.*

The proof uses an o-minimal structure on the cover as well as the theorem by Bays-Hart-Hyttinen-Kesala-Kirby.

A weaker result is also proved without restriction on $\dim \mathbb{X}$.

What is mathematical meaning of the categoricity statements?

Theorem(2023) *Let* $\mathbb{X}_a$ *be an abelian variety or a Shimura variety over* $\mathbb{Q}(a)$. *Let* $\sigma \in \mathrm{Aut}\,\mathbb{C}$.
Suppose there is $\Sigma(\mathbb{X}_a)$, *a categorical* $L_{\omega_1,\omega}$*-sentence for cover of* $\mathbb{X}_a$ *and suppose* $\Sigma(\mathbb{X}_a^\sigma)$ *is such a sentence for* $\mathbb{X}_a^\sigma$, *conjugation by the automorphism* σ. *Then*

$$\Sigma(\mathbb{X}_a^\sigma) = \Sigma(\mathbb{X}_a)^\sigma \quad \Rightarrow \quad \mathbb{X}_a^\sigma \cong \mathbb{X}_a \text{ or } \mathbb{X}_a^\sigma \cong \mathbb{X}_a^{cc}$$

(biholomorphic iso, complex conjgation ^{cc}*)*

Corollary. $\Sigma(\mathbb{X}_a)$ is a **complete** topological invariant of the complex variety $\mathbb{X}_a$.

A complementary problem

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Let $\mathbb{X} = \mathbb{X}(\mathbb{C})$ be a complex algebraic variety, $\sigma \in \mathrm{Aut}\,\mathbb{C}$.
Suppose there exists $\Sigma(\mathbb{X})$, a categorical $L_{\omega_1,\omega}$-sentence for the cover structure of $\mathbb{X}$.

Does a categorical $L_{\omega_1,\omega}$-sentence $\Sigma(\mathbb{X}^\sigma)$ for the cover structure of $\mathbb{X}^\sigma$ exist? How one obtains $\Sigma(\mathbb{X}^\sigma)$ from $\Sigma(\mathbb{X})$?

Examples: Topological and discrete invariants of conjugated algebraic varieties

- Serre's example of Galois conjugated varieties with different π_1 (i.e. the Galois type of $\mathbb{X}_a$ is not a complete topological invariant)

- Kazhdan theorem: a Galois conjugation of an arithmetic variety is arithmetic ($\mathbb{X} = \Gamma(\mathbb{Z})\backslash\mathcal{H}$)

- Langlands' conjecture on conjugated Shimura varieties gives an (incomplete?) list of invariants $\Sigma'(\mathbb{X}_a)$ and describes the transformation

$$\Sigma'(\mathbb{X}_a) \quad\longrightarrow\quad \Sigma'(\mathbb{X}_a^\sigma)$$

What Langlands says

Automorphic representations, Shimura varieties, and motives, Ein Märchen(1979)
The problem of conjugation is formulated in the sixth section as a conjecture, which was arrived at only after a long sequence of revisions. My earlier attempts were all submitted to Rapoport for approval, and found lacking. They were too imprecise, and were not even in principle amenable to proof by Shimuras methods of descent.