Tame pairs of transseries fields

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Transseries

Let $\mathbb{T}$ be the differential field of logarithmic-exponential transseries constructed by van den Dries–Macintyre–Marker. Example series in $\mathbb{T}$:

$$7e^{e^x+e^{x/2}+e^{x/4}+\cdots} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 42 + x^{-1} + x^{-2} + \cdots + e^{-x}$$

Remarks:

- $\mathbb{T}$ is a real closed field equipped with a derivation.
- $\mathbb{T}$ is exponentially bounded.

Model completeness for transseries

Let $\mathcal{O}_{\mathbb{T}} := \mathrm{conv}_{\mathbb{T}}(\mathbb{R}) = \{a \in \mathbb{T} : |a| \leqslant r \text{ for some } r \in \mathbb{R}^{>0}\}$.

Theorem (Aschenbrenner–van den Dries–van der Hoeven)

$(\mathbb{T}, \mathcal{O}_{\mathbb{T}})$ *is model complete and completely axiomatized by the theory* $T^{\mathrm{nl}}_{\mathrm{small}}$ *of H-fields with small derivation that are newtonian, ω-free, and Liouville closed.*

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Remarks on the language:

- Here, $\mathbb{T}$ is construed as an (ordered) differential field and $(\mathbb{T}, \mathcal{O}_{\mathbb{T}})$ is its expansion to an ordered valued differential field.
- Since $\mathbb{R} = C_{\mathbb{T}} := \{a \in \mathbb{T} : \partial a = 0\}$, $\mathcal{O}_{\mathbb{T}}$ is existentially definable in $\mathbb{T}$ as a differential field. In particular, $T^{\mathrm{nl}}_{\mathrm{small}}$ can be formulated in the language of differential fields.
- But careful! $\mathbb{T}$ is not model complete without the valuation ring.

Beyond transseries: Hyperseries

$E(x+1) = \exp E(x)$ has no solution in $\mathbb{T}$ but has a solution that belongs to a Hardy field (Boshernitzan).

Bagayoko–van der Hoeven–Kaplan construct a field $\mathbb{H} \supseteq \mathbb{T}$ (also denoted by $\widetilde{\mathbb{L}}$) of hyperseries that is closed under hyperexponentials $\exp_\alpha$ and hyperlogarithms $\log_\alpha$ for all ordinals α.

Let $\mathcal{O}_{\mathbb{H}} = \mathrm{conv}_{\mathbb{H}}(\mathbb{R})$.

Theorem (Bagayoko)

$(\mathbb{H}, \mathcal{O}_{\mathbb{H}}) \succcurlyeq (\mathbb{T}, \mathcal{O}_{\mathbb{T}})$. *In particular,* $(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ *is model complete.*

A second valuation

On $\mathbb{H}$, we have its natural valuation ring $\mathcal{O}_{\mathbb{H}}$. But $\dot{\mathcal{O}} = \operatorname{conv}_{\mathbb{H}}(\mathbb{T}) \supsetneq \mathcal{O}_{\mathbb{H}}$ is another valuation ring.

Comparing the valuations:

- $a \prec b \iff |a/b| < r$ for every $r \in \mathbb{R}^{>0}$
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For example:

- $x \prec \exp(x)$
- $x \mathrel{\dot{\not\prec}} \exp(x)$
- $x \mathrel{\dot{\prec}} \exp_\omega(x)$

So with both $\mathcal{O}_{\mathbb{H}}$ and $\dot{\mathcal{O}}$ we can express more about $\mathbb{H}$.

Model completeness for hyperseries

Recall (B): $(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ is model complete.

Theorem (PC)

There is $(\mathbb{T}, \mathcal{O}_{\mathbb{T}}) \preccurlyeq (\mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*}) \preccurlyeq (\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ *with* $\mathrm{conv}_{\mathbb{H}}(\mathbb{T}^*) = \mathrm{conv}_{\mathbb{H}}(\mathbb{T})$ *such that* $(\mathbb{H}, \mathrm{conv}_{\mathbb{H}}(\mathbb{T}^*), \mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$ *is model complete.*

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Question: Can we give an explicit description of such a $\mathbb{T}^*$?

Remark: Similar results hold for **No** or maximal Hardy fields.

Why not $(\mathbb{H}, \mathbb{T})$? The problem is that $\mathbb{T}$ is too small: e.g., $\log_\alpha(x)$ for infinite α is exponentially bounded but has no standard part in $\mathbb{T}$.

Tame pairs of real closed fields

Definition

Let $L \subsetneq K$ be RCFs with $\mathcal{O} = \mathrm{conv}_K(L)$ and $\smallo = \{a \in K : |a| < L^{>0}\}$, the maximal ideal of $\mathcal{O}$. Then L is **tame in** K if $\mathcal{O} = L + \smallo$, i.e., every element of $\mathcal{O}$ has a (unique) "standard part" in L.

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Example

1. $(\mathbb{T}, \mathbb{R})$
2. $(\bigcup_{n>0} \mathbb{R}(\!(t^{1/n})\!), \mathbb{R})$

Tame pairs go back to Macintyre and Cherlin–Dickmann; enrichments of these structures by van den Dries–Lewenberg.

Transserial tame pairs

Definition

A **transserial tame pair** is a pair (K, L) of differential fields such that:

1. $K, L \models T^{\mathrm{nl}}_{\mathrm{small}}$ as differential fields;
2. $L \subsetneq K$ is a proper differential subfield of K;
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Although $\mathbb{T}$ is not model complete without the valuation ring:

Lemma

Let (K, L) be a transserial tame pair. Then $C_L = C_K$ and so $(L, \mathcal{O}_L) \preccurlyeq (K, \mathcal{O}_K)$, where $\mathcal{O}_K = \mathrm{conv}_K(C_K)$.

Model completeness for transserial tame pairs

Theorem (PC)

The theory of transserial tame pairs is model complete in the language $\{+, -, \cdot, 0, 1, \leqslant, \partial, P, \mathcal{O}_P, \dot{\mathcal{O}}\}$.

In a transserial tame pair (K, L), interpret:

- P as L;
- $\mathcal{O}_P$ as $\mathrm{conv}_L(C_L)$;
- $\dot{\mathcal{O}}$ as $\mathrm{conv}_K(L)$.

Application to $\mathbb{H}$

Proposition

Let $K, L \models T^{\mathrm{nl}}_{\mathrm{small}}$ with $L \subsetneq K$. With $\dot{\mathcal{O}} := \mathrm{conv}_K(L)$ and $\dot{o}$ its maximal ideal, $\dot{K} := \dot{\mathcal{O}}/\dot{o}$ is the residue field of $(K, \dot{\mathcal{O}})$ and satisfies $\dot{K} \models T^{\mathrm{nl}}_{\mathrm{small}}$.

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There is $(\mathbb{T}, \mathcal{O}_{\mathbb{T}}) \preccurlyeq (\mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*}) \preccurlyeq (\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ with
$\mathrm{conv}_{\mathbb{H}}(\mathbb{T}^*) = \mathrm{conv}_{\mathbb{H}}(\mathbb{T}) = \dot{\mathcal{O}}$ *such that $(\mathbb{H}, \dot{\mathcal{O}}, \mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$ is model complete.*

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Proof.

As $(\mathbb{H}, \dot{\mathcal{O}})$ is differential-henselian, by ADH we can extend $\mathbb{T}$ to a lift $\mathbb{T}^* \subseteq \dot{\mathcal{O}}$ of $\dot{K} \models T^{\mathrm{nl}}_{\mathrm{small}}$. Then $\mathbb{T}^*$ is tame in $\mathbb{H}$, so $(\mathbb{H}, \dot{\mathcal{O}}, \mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$ is a transserial tame pair and thus model complete. $\square$

Differential-Hensel-Liouville closed pre-H-fields

Key step: Develop model theory of structures like $(K, \dot{\mathcal{O}})$.

Let T^{dhl} be the theory of pre-H-fields that are differential-henselian and Liouville closed with nontrivial valuation.

Recall (ADH): $\mathbb{T}$ is completely axiomatized by the theory $T^{\mathrm{nl}}_{\mathrm{small}}$ of H-fields with small derivation that are newtonian, ω-free, and Liouville closed.

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Theorem (PC)

$(K, \dot{\mathcal{O}}) \models T^{\mathrm{dhl}}$ and $\dot{K} \models T^{\mathrm{nl}}_{\mathrm{small}}$ if and only if $(K, \dot{K})$ is a transserial tame pair (identifying $\dot{K} = \dot{\mathcal{O}}/\dot{o}$ with a lift inside $\dot{\mathcal{O}}$).

Idea for $\mathbb{H}$: Decomposes $\mathbb{H}$ into transexponential part $(\mathbb{H}, \dot{\mathcal{O}})$ and exponentially bounded part $\mathbb{T}^*$.

Interlude: Hahn fields

Proposition

No model of T^{dhl} is spherically complete (i.e., isomorphic to a Hahn field).

Proof.

Use the logarithmic derivative map and integration to obtain a surjective logarithmic cross-section, which cannot exist on a spherically complete ordered valued field by a theorem of Kuhlmann–Kuhlmann–Shelah. □

Relative results

Setup:

- Suppose that $(K, \dot{\mathcal{O}}) \models T^{\mathrm{dhl}}$.
- Identify $(\dot{K}, \dots)$ with a lift of the differential residue field of $(K, \dot{\mathcal{O}})$, possibly expanded with additional structure.
- Expand with a standard part map $\pi \colon \dot{\mathcal{O}} \to \dot{K}$.

Define K^* likewise.

Theorem (PC)

1. $(K, \dot{\mathcal{O}}, \dot{K}, \dots, \pi) \equiv (K^*, \dot{\mathcal{O}}^*, \dot{K}^*, \dots, \pi^*) \iff (\dot{K}, \dots) \equiv (\dot{K}^*, \dots)$.
2. *If $(\dot{K}, \dots)$ is model complete, then so is $(K, \dot{\mathcal{O}}, \dot{K}, \dots, \pi)$.*

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2. *If $(\dot{K}, \dots)$ is model complete, then so is $(K, \dot{\mathcal{O}}, \dot{K}, \dots, \pi)$.*

Questions:

- Relative quantifier elimination?
- What is the structure induced on $\dot{K}$?
- Further applications to hyperseries (or **No** or maximal Hardy fields)?
- What about adding a (trans)exponential function?

Thank you!

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