Surreal ordered exponential fields

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The surreal numbers

The class $\mathbf{No}$ of surreal numbers are generated as follows:

Construction

If L and R are two sets of surreal numbers and no member of L is $\geqslant$ any member of R, then $\{L \mid R\}$ is a surreal number.

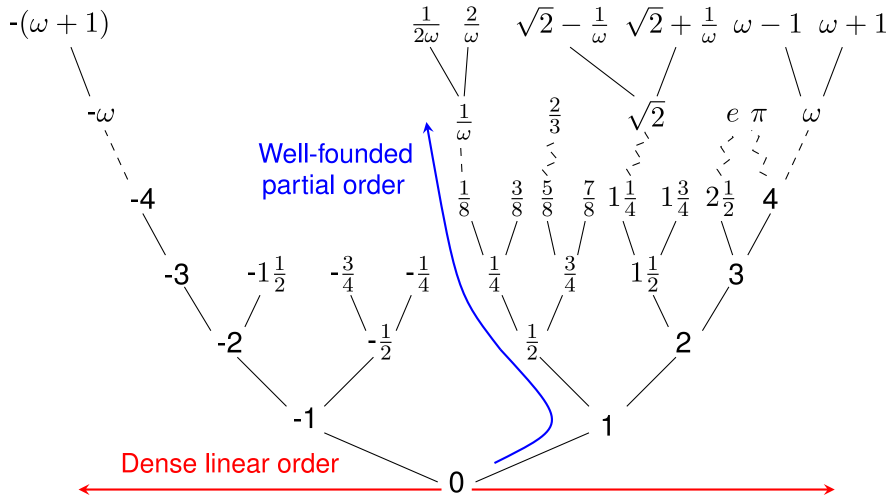
The *simplest* surreal number is $0 = \{\ \mid\ \}$. After constructing 0, we can construct $1 = \{0 \mid\ \}$ and $-1 = \{\ \mid 0\}$.

We use $\{L \mid R\}$ to denote the *simplest* number lying between L and R, so $\{-1 \mid 1\} = 0$ has already been constructed. Using our numbers 0, 1, and -1, we can construct four *new* numbers:

$$-2 := \{\ \mid -1\}, \quad -\frac{1}{2} := \{-1 \mid 0\}, \quad \frac{1}{2} := \{0 \mid 1\}, \quad 2 := \{1 \mid\ \}.$$

The surreal number tree

The surreal numbers are best visualized as a tree:



Adding and multiplying surreal numbers

Given a surreal number $x = \{L \mid R\}$, we use x^L to denote a typical element of L, and x^R to denote a typical element of R. Addition and multiplication can be defined recursively as follows:

$$x + y \ := \ \{x^L + y,\ x + y^L \mid x^R + y,\ x + y^R\}.$$

The idea is that $x^L < x$, so $x^L + y < x + y$, and so on.

$$xy \ := \ \left\{ \begin{matrix} x^L y + x y^L - x^L y^L, \\ x^R y + x y^R - x^R y^R \end{matrix} \ \middle| \ \begin{matrix} x^L y + x y^R - x^L y^R, \\ x^R y + x y^L - x^R y^L \end{matrix} \right\}.$$

Since $x - x^L, y^R - y > 0$, we should have

$$(x - x^L)(y^R - y) \ = \ x^L y + x y^R - x^L y^R - xy \ > \ 0,$$

and so $xy < x^L y + x y^R - x^L y^R$.

Surreal exponentiation

Gonshor defined an *exponential function* on the surreals, that is, an ordered group isomorphism $\exp\colon \mathbf{No} \to \mathbf{No}^{>}$.

We may define $\exp x$ recursively by

$$\left\{0,\ (\exp x^L)[x - x^L]_n,\ (\exp x^R)[x - x^R]_{2n+1} \;\middle|\; \frac{\exp x^L}{[x^L - x]_{2n+1}},\ \frac{\exp x^R}{[x^R - x]_n}\right\},$$

where $[y]_n := \sum_{k \leqslant n} \frac{y^k}{k!}$, and $[y]_{2n+1}$ is only included when it is positive.

Theorem (van den Dries-Ehrlich, 2001)

The surreal ordered exponential field is an elementary extension of the real ordered exponential field.

The motivating question

A subclass $X \subseteq \mathbf{No}$ is said to be **initial** if it is downward-closed under the well-founded partial order $<_s$.

An **ordered logarithmic field** is an ordered field K with an ordered group *embedding* $\log\colon K^> \to K$.

If this embedding is *surjective*, then we call K an **ordered exponential field** and denote the inverse of $\log$ by $\exp\colon K \to K^>$.

In our paper *Surreal ordered exponential fields*, Philip Ehrlich and I considered the following question:

Question

Which ordered exponential fields are isomorphic to initial exponential subfields of $\mathbf{No}$*?*

Before giving an answer, I'll briefly discuss the analogous question for *ordered fields*, which was answered by Ehrlich in 2001.

Hahn fields

Let Γ be an ordered abelian group (possibly a proper class). The **Hahn field** $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ consists of all transfinite series $\sum_{\beta<\alpha} r_\beta t^{\gamma_\beta}$, where $(\gamma_\beta)_{\beta<\alpha}$ is a decreasing sequence in Γ and each r_β is in $\mathbb{R} \setminus \{0\}$.

A **truncation** of $\sum_{\beta<\alpha} r_\beta t^{\gamma_\beta} \in \mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ is an element of the form $\sum_{\beta<\alpha_0} r_\beta t^{\gamma_\beta}$ for some $\alpha_0 \leqslant \alpha$. The **cross-section** of $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ is the multiplicative group t^{Γ}.

Theorem (Conway, 1976)

$\mathbb{R}((t^{\mathbf{No}}))_{\mathbf{On}}$ *is isomorphic to* $\mathbf{No}$*, via a map sending* t *to* ω*.*

Thus, we may represent each $x \in \mathbf{No}$ as a series $x = \sum_{\beta<\alpha} r_\beta \omega^{\gamma_\beta}$. We sometimes write $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}$.

Initial subfields of $\mathbf{No}$

Let K be a subfield of $\mathbf{No}$. Then $K \subseteq \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}$, so take Γ smallest with $K \subseteq \mathbb{R}((\omega^{\Gamma}))_{\mathbf{On}}$. Suppose K is initial. Then:

- $\sum_{\beta<\alpha_0} r_\beta \omega^{\gamma_\beta} \leqslant_s \sum_{\beta<\alpha} r_\beta \omega^{\gamma_\beta}$ for any $\alpha_0 \leqslant \alpha$, so K is *truncation closed*, i.e. any truncation of $x \in K$ belongs to K.
- Suppose $\sum_{\beta<\alpha} r_\beta \omega^{\gamma_\beta} \in K$ and let $\beta_0 < \alpha$. Then $\sum_{\beta<\beta_0} r_\beta \omega^{\gamma_\beta}$ and $\sum_{\beta_0 \leqslant \beta<\alpha} r_\beta \omega^{\gamma_\beta}$ belong to K. Since $\omega^{\gamma_{\beta_0}} \leqslant_s \sum_{\beta_0 \leqslant \beta<\alpha} r_\beta \omega^{\gamma_\beta}$, we see that $\omega^{\gamma_{\beta_0}} \in K$. Thus, K is *cross-sectional*, i.e. $\omega^{\Gamma} \subseteq K$.
- It follows that Γ is an initial sub*group* of $\mathbf{No}$.

This turns out to be enough:

Theorem (Ehrlich, 2001)

A subfield $K \subseteq \mathbf{No}$ is initial if and only if it is a truncation closed, cross-sectional subfield of $\mathbb{R}((\omega^{\Gamma}))_{\mathbf{On}}$ for some initial subgroup $\Gamma \subseteq \mathbf{No}$.

More on initial subfields

Corollary

An ordered field K is isomorphic to an initial subfield of $\mathbf{No}$ if and only if it is isomorphic to a truncation closed, cross-sectional subfield of $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$, where Γ is isomorphic to an initial ordered subgroup of $\mathbf{No}$.

Explicitly, let $K \subseteq \mathbb{R}((t^\Gamma))_{\mathbf{On}}$ be truncation closed and cross-sectional, let $\iota\colon \Gamma \to \mathbf{No}$ be an initial ordered group embedding, and let ι^* be the map:

$$\sum_{\beta<\alpha} r_\beta t^{\gamma_\beta} \mapsto \sum_{\beta<\alpha} r_\beta \omega^{\iota(\gamma_\beta)} \colon \mathbb{R}((t^\Gamma))_{\mathbf{On}} \to \mathbf{No}.$$

Then $\iota^*(K)$ is initial.

Corollary

An initial map ι always exists if Γ is divisible, so any real closed ordered field initially embeds into $\mathbf{No}$ by Mourgues-Ressayre.

Exponential subfields of $\mathbf{No}$

It follows that the initial exponential subfields of $\mathbf{No}$ are exactly the truncation closed, cross-sectional subfields of $\mathbb{R}((\omega^{\Gamma}))_{\mathbf{On}}$, where Γ is an initial subgroup of $\mathbf{No}$. This is not a very satisfying answer.

Using the identification $\mathbf{No} \simeq \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}$, we can give a nicer description of $\exp$ in terms of its restrictions:

- $\exp$ maps $\mathbb{R}((\omega^{\mathbf{No}^{>}}))_{\mathbf{On}}$, the *purely infinite elements*, onto $\omega^{\mathbf{No}}$.
- $\exp$ restricts to the real exponential on $\mathbb{R} \subseteq \mathbf{No}$.
- For $\varepsilon \in \mathbf{No}^{\prec}$, the class of *infinitesimal elements*, we have

$$\exp \varepsilon \ = \ \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \ \in \ 1 + \mathbf{No}^{\prec}.$$

- As $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}^{>}}))_{\mathbf{On}} \oplus \mathbb{R} \oplus \mathbf{No}^{\prec}$, this determines $\exp$.

An initial guess

A **logarithmic Hahn field** is a Hahn field $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ equipped with an ordered group embedding $\log\colon \mathbb{R}((t^\Gamma))^{>}_{\mathbf{On}} \to \mathbb{R}((t^\Gamma))_{\mathbf{On}}$ where:

- $\log x \leqslant x - 1$ for all $x \in \mathbb{R}((t^\Gamma))^{>}_{\mathbf{On}}$;
- $\log$ maps t^Γ into $\mathbb{R}((t^{\Gamma^>}))_{\mathbf{On}}$;
- $\log$ restricts to the real logarithm on $\mathbb{R}^>$;
- If ε is infinitesimal, then

$$\log(1+\varepsilon) \;=\; \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varepsilon^k}{k}.$$

We may naively guess that an ordered exponential field is isomorphic to an initial exponential subfield of $\mathbf{No}$ if and only if it is isomorphic to a truncation closed, cross-sectional exponential subfield of a logarithmic Hahn field.

What goes right?

We use the following approach, pioneered by Ressayre (1993) and van den Dries-Macintyre-Marker (1994).

Let K be a truncation closed, cross-sectional exponential subfield of a logarithmic Hahn field $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$. Let K_0 be a truncation closed logarithmic subfield of K, and assume that

$$\sum_{\beta<\alpha} r_\beta t^{\gamma_\beta} \in K_0 \implies t^{\gamma_\beta} \in K_0 \text{ for all } \beta.$$

Assume we have an initial logarithmic field embedding $\iota\colon K_0 \to \mathbf{No}$ that preserves monomials and infinite sums.

- If $x = \sum_{\beta<\alpha} r_\beta t^{\gamma_\beta} \in K$, α is a limit ordinal, and every proper truncation of x is in K_0, then ι can be extended to include x.
- If $x \in K^>$ and $\log x \in K_0$, then ι can be extended to include x.

What's missing?

Assume K_0 is maximal with respect to the previous extensions and let $x = t^\gamma \in K \setminus K_0$. Define $(x_n)_{n\in\mathbb{N}}$ as follows:

$$x_0 := x, \qquad x_{n+1} := |\log x_n - a_n|$$

where a_n is the maximal truncation of $\log x_n$ in K_0.

Let $y := \left\{\iota(K_0^{<x}) \mid \iota(K_0^{>x})\right\}$ and set

$$y_0 := y, \qquad y_{n+1} := |\log y_n - \iota(a_n)|.$$

Fact

Under mild assumptions, $y_n \in \omega^{\mathbf{No}}$ *for each* n.

Transseries fields

Definition (Schmeling, 2001)

A ***transseries field*** *is a logarithmic Hahn field* $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ *such that for all sequences* $(\gamma_n)_{n\in\mathbb{N}}$ *in* Γ *and* $(a_n)_{n\in\mathbb{N}}$ *in* K*, if* a_n *is a truncation of* $\log t^{\gamma_n}$ *and* $\log t^{\gamma_n} - a_n = rt^{\gamma_{n+1}} + \cdots$*, then* $\log t^{\gamma_n} - a_n = \pm t^{\gamma_{n+1}}$ *for* n *sufficiently large.*

If $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ is a transseries field and $(x_n)_{n\in\mathbb{N}}$ is as above, then $x_n \in t^\Gamma$ for n sufficiently large.

Theorem (Ehrlich-K., 2021)

An ordered exponential field K *is isomorphic to an initial exponential subfield of* $\mathbf{No}$ *if and only if it is isomorphic to a truncation closed, cross-sectional subfield of a transseries field* $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$*.*

Elementary extensions of $\mathbb{R}_{\text{an,exp}}$

Fact (van den Dries-Macintyre-Marker, 1994)

Any Hahn field $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ *with* Γ *divisible can be expanded to an elementary extension of* $\mathbb{R}_{\text{an}}$*, the real field with restricted analytic functions. This is done using Taylor expansion.*

Theorem (Ehrlich-K., 2021)

Any elementary extension of $\mathbb{R}_{\text{an,exp}}$ *admits a truncation closed, cross-sectional exponential field embedding into a transseries field* $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ *that preserves restricted analytic functions.*

Corollary (First shown by Fornasiero, 2013)

Any elementary extension of $\mathbb{R}_{\text{an,exp}}$ *admits an initial elementary embedding into the surreals.*

Models of real exponentiation

The same holds when restricted analytic functions are replaced with any *Weierstrass system* that includes the restricted exponential.

Open Question

Let $K \models \mathrm{Th}(\mathbb{R}_{\exp})$*. Does* K *admit an initial embedding into* $\mathbf{No}$*?*

The obvious approach is to use an embedding result by Ressayre (1993), which gives a truncation closed, cross-sectional field embedding ι of any such K into a Hahn field.

The issue is that for ε infinitesimal, it may not happen that

$$\iota(\log(1+\varepsilon)) \;=\; \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\iota(\varepsilon)^k}{k}.$$

This is really the only obstruction.

Transserial embeddings

In proving the main theorem, we show the following:

Corollary (First shown by Berarducci-Mantova, 2018)

The surreals are a transseries field.

Let $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ be a transseries field. An embedding $\mathbb{R}((t^\Gamma))_{\mathbf{On}} \to \mathbf{No}$ is called **transserial** if it preserves logarithms, infinite sums, products, and monomials.

Open Question

Which transseries fields admit initial transserial embeddings into $\mathbf{No}$*?*
Which logarithmic fields are isomorphic to initial logarithmic subfields of $\mathbf{No}$*?*

Partial answers

Looking at the main theorem differently, we see that any transseries field that has a truncation closed, cross-sectional exponential subfield admits an initial transserial embedding into $\mathbf{No}$.

Corollary

Any transseries field admits a truncation closed transserial embedding into $\mathbf{No}$*.*

Proof.

Schmeling showed that any transseries field $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ extends to a transseries field $\mathbb{R}((t^{\Gamma^*}))_{\mathbf{On}}$ that is closed under exponentiation. Any initial transserial embedding $\mathbb{R}((t^{\Gamma^*}))_{\mathbf{On}} \to \mathbf{No}$ induces a truncation closed embedding $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}} \to \mathbf{No}$. □

Logarithmic-exponential transseries and derivations

Let $\mathbb{T}$ be the field of logarithmic-exponential transseries. There is a canonical embedding $\mathbb{T} \to \mathbf{No}$ sending x to ω.

This is even an *elementary embedding of differential fields*, with the derivation on $\mathbf{No}$ as defined by Berarducci-Mantova (2018).

Theorem (Ehrlich-K., 2021)

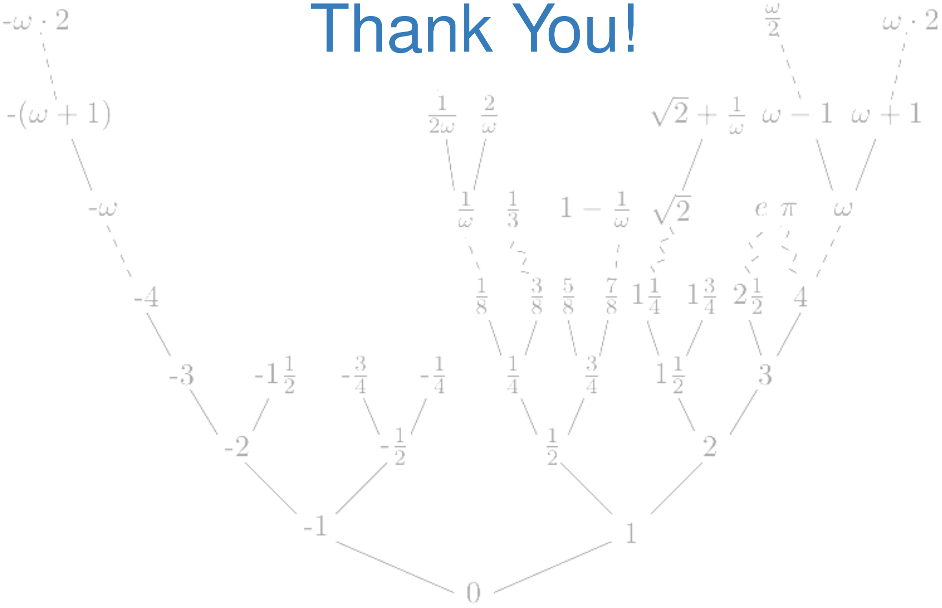
The image of the canonical embedding $\mathbb{T} \to \mathbf{No}$ is initial.

Open Question

Which ordered differential fields admit initial embeddings into $\mathbf{No}$?

This question is difficult. There are many possible derivations on $\mathbf{No}$, and while the theory of $\mathbf{No}$ as a differential field is understood thanks to Aschenbrenner-van den Dries-van der Hoeven (2017 and 2019), it is still quite complicated.

Thank You!



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