# Towards Quasiminimality and Exponential Algebraic Closedness

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# Abstract


## Towards Quasiminimality and Exponential Algebraic Closedness

I will explain the notions of quasiminimality and exponential algebraic closedness, and survey some progress towards proving that they hold in the complex exponential field.
I will also survey some other related work.

# Outline

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# Zilber's Quasiminimality Conjecture

Recall that definable subsets of $\mathbb{C}_{\text{field}}$ are finite or cofinite. We say $\mathbb{C}_{\text{field}}$ is minimal.
In $\mathbb{C}_{\exp}$, we have $\mathbb{Z}$ defined by

$$x \in \mathbb{Z} \text{ iff } \forall y[\exp(y) = 1 \implies \exp(xy) = 1]$$

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## Open Question

Is $\mathbb{R}$ a definable subset of $\mathbb{C}_{\exp}$?

If true, every continuous function $\mathbb{C} \to \mathbb{C}$ is definable in $\mathbb{C}_{\exp}$, and definable sets have no geometric meaning – the subject is descriptive set theory.

## Question

For which complex functions $f$ is $\langle \mathbb{C}; +, \cdot, f \rangle$ quasiminimal?

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## Examples

- complex conjugation – $\mathbb{R}$ definable so o-minimal, not quasiminimal
- *j*-function – domain is $\mathbb{H}$ or $\mathbb{H}^+ \cup \mathbb{H}^- = \mathbb{C} \setminus \mathbb{R}$

## Generic functions (Dmitrieva in progress)

"Most" entire functions are quasiminimal.
In particular the Liouville functions defined by Wilkie and shown by him and Koiran to satisfy Zilber's first-order theory of a generic function
Also work of Le Gal on strongly transcendental functions

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## Examples

- Weierstrass $\wp$-functions – seem similar to $\exp$
- Exponential maps of abelian varieties $\exp_A : \mathbb{C}^g \to A(\mathbb{C})$. – seem similar again
- Bieberbach–Fatou example 1920s $f : \mathbb{C}^2 \to \mathbb{C}^2$ image open not dense – not QM.

## Questions

Koiran: Is the expansion of $\mathbb{C}$ by all 1-variable entire functions quasiminimal?
Dmitrieva: What about the expansion by all 1-variable meromorphic functions?

# Some Quasiminimality theorems

## Theorem (Bays, K 2018)

*If $\mathbb{C}_{\exp}$ is exponentially algebraically closed (EAC) then it is quasiminimal.*
*Similar results for other expansions of $\mathbb{C}$ with known Ax–Schanuel theorem.*

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Using that technique:

## Theorem (K 2019)

*Let $\Gamma = \{(z, \exp(z + q + 2\pi i r)) \mid z \in \mathbb{C}, q, r \in \mathbb{Q}\}$.*
*Then the blurred exponential field $\langle \mathbb{C}; +, \cdot, \Gamma \rangle$ is quasiminimal.*

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With Dmitrieva, we hope to push this technique further.

A variant of the technique, plus Gallinaro's proof of the relevant analogue of EAC gave:

## Theorem (Gallinaro, K 2023)

*For* $\lambda \in \mathbb{C}$, *let* $\Gamma_\lambda = \{(\exp(z), \exp(\lambda z)) \mid z \in \mathbb{C}\}$, *the graph of the multivalued map* $w \mapsto w^\lambda$. *Then the structure* $\mathbb{C}$ *with complex powers*

$$\langle \mathbb{C}; +, \cdot, -, 0, 1, (\Gamma_\lambda)_{\lambda \in \mathbb{C}} \rangle$$

*is quasiminimal.*

# Outline

# Exponential fields – basics

## Definition

An exponential field (E-field) is a field $F$ of characteristic 0 equipped with a homomorphism $\exp : \langle F; +, 0\rangle \to \langle F^{\times}; \times, 1\rangle$.
It has standard kernel if $\ker(\exp) = \tau\mathbb{Z}$ with $\tau$ transcendental.
It is an EA-field (ELA-field) if it is algebraically closed (and $\exp$ is surjective).

If $F$ has standard kernel then $\mathbb{Z}$, $\mathbb{Q}$ are definable.
$L_{\omega_1,\omega}$-axiomatizable / omitting type of non-standard integer

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## Schanuel's conjecture / Schanuel property

For any $a_1, \ldots, a_n \in \mathbb{C}$ (or $F$), $\mathbb{Q}$-linearly independent, then

$$\mathrm{td}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \geqslant n.$$

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Axiomatizable by first-order scheme, modulo standard kernel:
for $V \subseteq \mathbb{C}^{2n}$ defined over $\mathbb{Q}$ with $\dim V < n$, take

$$\forall \bar{x}[(\bar{x}, e^{\bar{x}}) \in V \to \bar{x} \text{ is } \mathbb{Q}\text{-linearly dependent}]$$

## Theorem (K, Zilber 2014)

*Assuming Zilber's Conjecture on Intersections with Tori (multiplicative Zilber-Pink), the Schanuel Property relativized over the kernel is first-order axiomatizable.*

# Strong extensions

## Definition

*Given $A \subseteq F$, and a tuple $b$ from $F$, the relative predimension is*

$$\delta(b/A) := \mathrm{td}(b, \exp(b)/A, \exp(A)) - \mathrm{ldim}_{\mathbb{Q}}(b/A).$$

*An extension $A \subseteq F$ of exponential fields is strong if for all tuples $b \in F$, $\delta(b/A) \geqslant 0$, and the kernel of the exponential function does not extend. Notation: $A \lhd F$.*

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The strong extensions are those which preserve the Schanuel property.

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The strong extensions are those which preserve the Schanuel property.

## Theorem (Uniqueness of ELA-closure)

*A finitely generated partial exponential field (or a finitely generated extension of a countable ELA-field) has a unique smallest strong extension to an ELA-field.*

So to classify finitely generated strong extensions, we can consider finitely generated extensions of ELA-fields.

# Strong extensions: classification 1

## Configuration for a finitely generated extension

$A$ an ELA-field, $A \subseteq B$ extension generated over $A$ by $b_1, \ldots, b_n$ and $e^{b_i/m}$ for $i = 1, \ldots, n$ and $m \in \mathbb{N}^+$.
Assume that the $b_i$ are $\mathbb{Q}$-linearly independent over $A$. If $a \in A$ and $r_i \in \mathbb{Z}$, then

$$\exp\left(a + \sum_{i=1}^{n} \frac{r_i}{m} b_i\right) = e^a \prod_{i=1}^{n} (e^{b_i/m})^{r_i}$$

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The isomorphism type of the extension is given by the algebraic type of the infinite tuple $(\bar{b}, \exp(\bar{b}/m))_{m \in \mathbb{Z}}$ over $A$. For $m \in \mathbb{N}^+$, let

$$I_m(\bar{b}) = \left\{ f \in F[\bar{X}, \bar{Y}_m, \bar{Y}_m^{-1}] \;\middle|\; f(\bar{b}/m, e^{\bar{b}/m}, e^{-\bar{b}/m}) = 0 \right\}$$

and let $V_m = \operatorname{spec} I_m$. For each $m, r$ the variety $V_{mr}$ is like an $r^{\text{th}}$ root of $V_m$. To specify the isomorphism type of the extension we have to specify the infinite sequence of varieties.

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## Theorem ("Thumbtack Lemma", from Kummer theory, due to Zilber)

*Given such a sequence of varieties there is $N \in \mathbb{N}^+$ such that if we specify $V_N$ then all the other varieties $V_m$ are uniquely determined. We say $V_N$ is Kummer-generic.*

# Strong extensions: classification 2

The $V$ which occur are free and rotund: suppose $(\bar{x}, \bar{y})$ is generic in $V$ over $A$.

- **Free** : $\bar{x}$ is $\mathbb{Q}$-linearly independent over $A$ and $\bar{y}$ is multiplicatively independent over $A$. So $\exp$ is well-defined and no new kernel elements.
- **Rotund** : $\dim V \geqslant n$, and similarly for $\mathbb{Q}$-linear projections. This ensures the extension is strong.

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Every free and rotund $V$ gives a strong extension.

## Corollary (Finite presentation for finitely generated extensions)

*Finitely generated strong ELA-field extensions are determined up to isomorphism by a single algebraic variety, $V$.*

## Corollary

*There are only countably many isomorphism types of finitely generated ELA-fields (with standard kernel), and only countably many isomorphism types of finitely generated strong extensions of a countable ELA-field.*

# Zilber's exponential field

## Theorem

*There is an amalgamation category consisting of*

- *The finitely generated ELA-fields with standard kernel, satisfying the Schanuel Property;*
- *Strong extensions*

*and hence there is a Fraïssé limit,* $\mathbb{F}$.

Zilber's exponential field $\mathbb{B}_{\mathrm{exp}}$ is a continuum sized elementary extension of $\mathbb{F}$ which is quasiminimal.

## Corollary (of finite presentation)

*Exponentially algebraic types in* $\mathbb{B}_{\mathrm{exp}}$ *of finite tuples over* $\emptyset$ *and over strong finitely generated ELA-subfields A are isolated by a single formula, of the form:*

$$(\exists x_2, \ldots, x_n)[(\bar{x}, e^{\bar{x}}) \in V \wedge \bar{x} \text{ is } \mathbb{Q}\text{-linearly independent over } A_0]$$

*where* $A_0$ *is a certain finite subset of A.*

# Zilber's Nullstellensatz and EAC

$F$ an exponential field. Given a suitable algebraic subvariety $V$ of $F^n \times (F^\times)^n$, we can define an extension exponential field $F_1 = F|V$ by adding a point $(a_1, \ldots, a_n) \in F_1^n$ such that $(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \in V$.

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## Zilber's Nullstellensatz

If $V$ is free and rotund then there is already $\mathbf{a} \in F^n$ such that $(\mathbf{a}, e^{\mathbf{a}}) \in V$.

## Conjecture (EAC conjecture)

Zilber's nullstellensatz holds for $\mathbb{C}_{\exp}$.

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## Zilber's Nullstellensatz

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## Conjecture (EAC conjecture)

Zilber's nullstellensatz holds for $\mathbb{C}_{\exp}$.

An exponential field $F$ is said to be exponentially-algebraically closed if it satisfies Zilber's nullstellensatz. Every exponential field $F$ can be embedded in an exponential-algebraic closure.

There is no reduction to exponential polynomials in one variable.

## Theorem (Bays, Kirby 2018)

*If the EAC conjecture is true then $\mathbb{C}_{\exp}$ is quasiminimal.*

# Progress towards EAC

## Question

Given $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$, free and rotund, is there $a \in \mathbb{C}^n$ such that $(a, e^a) \in V$?
We can reduce to the case $\dim V = n$.

- $n = 1$ (Marker)
- Let $W = \mathrm{pr}_{\mathbb{C}^n} V$. If $\dim W = n$ then yes, (Masser, Brownawell, also D'Aquino, Fornasiero, Terzo).
- Same condition, geometric proof (Aslanyan, K, Mantova).
- Analogous case for exponential maps of Abelian varieties, (Aslanyan, K, Mantova).
- $\dim W = 1$ (Mantova, Masser)
- Exponential sums / complex powers (Gallinaro)
- Other abelian and $j$-examples (Gallinaro)
- $j$-situation with $\dim W = n$ (Eterovic, Herrera)
- $\Gamma$-function with $\dim W = n$ (Eterovic, Padgett)
- Some examples with $j$ and its derivatives (Aslanyan, Eterovic, Mantova)

# Outline

# Exponential Algebraicity

Let $\langle F; +, \cdot, \exp\rangle$ be any exponential field (even ordered!)

## Definition

$a_1 \in F$ is exponentially algebraic over a subset $B$ in $F$ iff for some $n \in \mathbb{N}$ there are:

- $\bar{a} = (a_1, \ldots, a_n) \in F^n$
- polynomials $p_1, \ldots, p_n \in \mathbb{Z}[\bar{X}, e^{\bar{X}}, \bar{Y}]$
- $\bar{b}$ from $B$

such that setting $f_i(\bar{X}, \bar{Y}) = p_i(\bar{X}, e^{\bar{X}}, \bar{Y})$ we have

- $f_i(\bar{a}, \bar{b}) = 0$ for each $i = 1, \ldots, n$, and
- $\begin{vmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{vmatrix} (\bar{a}, \bar{b}) \neq 0.$

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Exponentially Transcendental over $B$ in $F \iff$ not exponentially algebraic over $B$ in $F$

$a_1 \in \mathrm{ecl}(B)$ depends on witnesses $a_2, \ldots, a_n$, so on the existential type of $a$ over $B$, not just the quantifier-free type.

## Theorem (K, 2010)

Exponential-algebraic closure, ecl, is a pregeometry on any exponential field.

# Exponential algebraicity, predimension, hull

## Theorem

*Suppose $F$ an exponential field, $B \lhd F$. Then $a_1 \in \mathrm{ecl}^F(B)$ if and only if there is $n \in \mathbb{N}$, $a_2, \ldots, a_n \in F$ such that $\delta(a_1, \ldots, a_n/B) = 0$.*
*These $a_2, \ldots, a_n$ are called the witnesses of exponential algebraicity and they are unique up to change of $\mathbb{Q}$-linear basis over $B$.*

## Definition

The hull $\lceil B \rceil$ of $B$ in $F$ is the smallest $\mathbb{Q}$-linear subspace of $F$ containing $B$ which is strong in $F$.
In the above case, $\lceil Ba_1 \rceil = \mathbb{Q}$-span of $Ba_1 \ldots a_n$.
We write $\lceil B \rceil^{\mathrm{EA}}$ and $\lceil B \rceil^{\mathrm{ELA}}$ for the smallest exponential subfields of $F$ containing $\lceil B \rceil$ which are relatively EA-closed / ELA-closed in $F$.

## Theorem (Aslanyan, Kirby, in preparation)

*Let $A \subseteq \mathbb{B}_{\exp}$. The model-theoretic algebraic closure is*

$$\mathrm{acl}(A) = \lceil A \cup \{\tau\} \rceil^{\mathrm{EA}}.$$

# Strong independence

## Definition (Strong independence)

Let $F$ be an ELA-field and $A, B, C \subseteq F$. We say that *$A$ is strongly independent from $B$ over $C$ in $F$*, and write $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\lhd, F} B$, if:

1. $\lceil AC \rceil^{\mathrm{ELA}} \mathop{\smile\!\!\!\!\!\!|}_{\lceil C \rceil^{\mathrm{ELA}}}^{\mathrm{td}} \lceil BC \rceil^{\mathrm{ELA}}$, and
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If $A, B, C$ are strong ELA-subfields of $F$ and $C \subseteq A \cap B$ then $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\triangleleft} B$ iff

1. $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\mathrm{td}} B$, and
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Under these conditions,

## Lemma

*$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\triangleleft} B$ is equivalent to $\lceil ABC \rceil_F^{\mathrm{ELA}}$ being a / the free amalgam of $A$ and $B$ over $C$ as ELA-fields.*

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## Theorem (Aslanyan, Henderson, Kamsma, K)

*Fix any ELA-field $K$, and let $\mathbf{ELAF}_{K,\triangleleft}$ be the category of strong ELA-extensions of $K$, with strong embeddings.*
*Then $\mathop{\smile\!\!\!\!\!\!|}^{\triangleleft}$ gives a stable independence relation on $\mathbf{ELAF}_{K,\triangleleft}$, so $\mathbf{ELAF}_{K,\triangleleft}$ is a stable AEC.*

* Technically we can only prove this either when $\ker(K)$ is sufficiently saturated, so not standard, or for syntactic types, not Galois types, over uncountable bases. Over countable bases it is fine.

# Outline

# Exponential sums equations

- In $\mathbb{C}_{\exp}$ we can express complicated equations like

$$e^{e^{\sin(z^2 - iz)}} + \cos(e^{z - 1/z}) + 1 = 0$$

In many applications, we do not iterate exponentiation but only use it to define complex powers: for fixed $\lambda \in \mathbb{C}$, define the multivalued $w^\lambda = \exp(\lambda \log w)$

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- Similarly, consider exponential sums: for fixed $\lambda_i \in \mathbb{C}$, and $w_j = \exp(z_j)$, $\exp\left(\sum_{j=1}^n \lambda_j z_j\right) = \prod_{j=1}^n w_i^{\lambda_j}$.
- More generally for a matrix $M = (\lambda_{ij})$, let $\mathbf{u} = M\mathbf{z}$ and $v_i = \exp u_i$.

## Definition

An exponential sums equation is an equation of the form $p(\mathbf{v}) = 0$, where $p \in \mathbb{C}[\mathbf{X}]$ and $\mathbf{v} = \exp(M\mathbf{z})$ as above.

A system of exponential sums equations gives an algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ of the form $L \times W$ where $L \subseteq \mathbb{C}^n$ is given by $\mathbb{C}$-linear equations and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic subvariety. A solution is a point $\mathbf{a} \in L$ such that $\exp(\mathbf{a}) \in W$.

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## Theorem (Gallinaro, 2022, Nullstellensatz for complex exponential sums)

*Suppose that $V = L \times W$ is a system of complex exponential sums equations which is free and rotund in the sense of exponential fields. Then there is a complex solution.*

# The complex powered field

## Definition

Let $\mathbb{C}^{\mathbb{C}}$ be the complex numbers considered as a $\mathbb{C}$-powered field, that is:

$$\mathbb{C}_{\mathbb{C}\text{-VS}} \xrightarrow{\exp} \mathbb{C}_{\text{field}}$$

where the codomain is $\mathbb{C}$ equipped with the field structure, the cover is $\mathbb{C}$ equipped only with its structure as a $\mathbb{C}$-vector space, and the covering map is the usual complex exponentiation.

The equations expressible in this structure with variables in the cover are exactly the exponential sums equations.

The expressible equations with variables in the field are the "$\mathbb{C}$-powered polynomial" equations.

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So there should be some reasonable geometric theory of algebraic geometry with complex powers.

# Almost all powers are generic

## Theorem (Gallinaro, Kirby, 2023)

*Let $K$ be a countable field of characteristic 0. Then up to isomorphism, there is exactly one $K$-powered field $\mathbb{E}^K$ of cardinality continuum which:*

(i) *has cyclic kernel,*

(ii) *satisfies the Schanuel property,*

(iii) *is $K$-powers closed, and*

(iv) *has the countable closure property.*

*Furthermore, it is quasiminimal.*

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If Schanuel's conjecture is true, the same holds for many other $\lambda$, including $\pi$, $2\pi i$.

## Theorem (Zilber, unpublished)

*The first-order theory of $\mathbb{E}^K$ is superstable.*