REAL ALGEBRA AND ITS INTERACTIONS WITH FUNCTIONAL ANALYSIS

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The slides of this talk are available at:
http://math.usask.ca/~skuhlman/SLIDESKG.pdf

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Positive Polynomials on Projective Limits of Algebraic Varieties

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Putinar's Striktpositivstellensatz

1. Quadratic modules

Let A be a commutative ring with 1 (assume $\mathbb{Q} \subseteq A$). A *quadratic module* Q is a subset of A such that $Q + Q \subset Q, 1 \in Q$ and $a^2 Q \subset Q$ for all $a \in A$.

We denote by $Q(S)$ the quadratic module generated in A by the set $S \subseteq A$. If S is finite, say $Q(S)$ is finitely generated (f.g.)

Throughout, a real algebraic affine variety $X \subseteq \mathbb{R}^d$ is the common zero set of a finite set of polynomials, and the algebra of regular functions on X is $A = \mathbb{R}[X] = \mathbb{R}[x_1, ..., x_d]/I(X)$, where $I(X)$ is the radical ideal associated to X.

The non-negativity set of $S \subseteq \mathbb{R}[X]$ is

$$K(S) = \{x \in X; f(x) \geq 0, \quad f \in S\}.$$

A quadratic module $Q \subseteq \mathbb{R}[X]$ is *archimedean* if for every element $f \in \mathbb{R}[X]$ there exists a positive scalar α such that $1 + \alpha f \in Q$. The non-negativity set of an archimedean quadratic module is always compact.

2. Positivstellensätze

The following Striktpositivstellensatz has attracted in the last decade a lot of attention from practitioners of polynomial optimization:

Theorem 2.1. *Let $Q \subseteq \mathbb{R}[X]$ be an archimedean quadratic module and assume that a polynomial f is positive on $K(Q)$. Then $f \in Q$.*

The aim of this talk is to extend the above Striktpositivstellensatz to more general convex cones of polynomials defined on real algebraic varieties. The motivation comes from the work of Kojima and Lasserre on sparse polynomials.

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Kojima-Lasserre Specialized Version

In his October 2006 Banff talk, Lasserre presented the following specialized version of Putinar's Positivstellensatz, which yields specialized SDP-relaxations for polynomial optimization with significant computational savings.

Let $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_m)$ and $Z = (Z_1, ..., Z_p)$.

Consider $K \subseteq \mathbb{R}^{n+m+p}$

$$K := \{(x, y, z) : (x, y) \in K_{xy}; (y, z) \in K_{yz}\}$$

$$K_{xy} := \{(x, y) \in \mathbb{R}^{n+m} : g_j(x, y) \geq 0, j \in J_{xy}\}$$

$$K_{yz} := \{(y, z) \in \mathbb{R}^{m+p} : h_j(y, z) \geq 0, j \in J_{yz}\}.$$

Let $f \in \mathbb{R}[X, Y] + \mathbb{R}[Y, Z]$.
(So in the definition of K and f there is no coupling of variables X and Z).
Can we have a specialized representation of $f > 0$ on K that preserves this coupling pattern?
If yes, can we extend to more than two subsets of variables (X, Y) and (Y, Z)?

Theorem 1 L (2006)
Assume that K is compact, and $Q(\{g_j\})) \subseteq \mathbb{R}[x, y], Q(\{h_j\}) \subseteq \mathbb{R}[y, z]$ are archimedean.
Then

$$f \in Q(\{g_j\})) + Q(\{h_j\}).$$

Let $\{I_1, I_2, ..., I_k\}$ be a finite covering of the set of indices $I = \{1, 2, ..., d\}$. Denote the polynomial ring in the variables $\{x_j : j \in I_j\}$ by $\mathbb{R}[x(I_j)]$ and the corresponding affine space by $\mathbb{R}^{I_j}$.

Assume that the covering $\{I_1, ..., I_k\}$ satisfies the *running intersection property* (RIP): for all $j, 2 \leq j \leq k$, there exists $k(j) < j$ such that

$$I_j \cap (I_1 \cup I_2 \cup ... \cup I_{j-1}) \subset I_{k(j)} \,.$$

Theorem 2 (Kojima-Lasserre)
Consider f.g. archimedean quadratic modules $Q_j \subseteq \mathbb{R}[x(I_j)], 1 \leq j \leq k$ with compact positivity sets $K(Q_j) \subseteq \mathbb{R}^{I_j}$. Let $\mathbf{K(Q_j)} = K(Q_j) \times \mathbb{R}^{I \setminus I_j}$ be the associated cylinders in $\mathbb{R}^I$. Consider the compact $\mathbf{K} := \cap_{\mathbf{j=1}}^{\mathbf{k}} \mathbf{K(Q_j)}$.
Let $f \in \mathbb{R}[x(I_1)] + ... + \mathbb{R}[x(I_k)]$ be strictly positive on $\mathbf{K}$. Then

$$f \in Q_1 + Q_2 + ... + Q_k \; .$$

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A Striktpositivstellensatz for fibre products.

Let $Z = X_1 \times_Y X_2$ be the (reduced) *fibre product* of affine real varieties. Specifically

$$f_i : X_i \longrightarrow Y, \quad i = 1, 2,$$

are given morphisms and

$$Z = \{(x_1, x_2) \in X_1 \times X_2;\ f_1(x_1) = f_2(x_2)\}.$$

This is still an algebraic variety, with the ring of regular functions

$$\mathbb{R}[X_1 \times_Y X_2] = \mathbb{R}[X_1] \otimes_{\mathbb{R}[Y]} \mathbb{R}[X_2].$$

Denote by $u_i : Z \longrightarrow X_i$, $i = 1, 2$, the projection maps, so that: $f_1 u_1 = f_2 u_2$.

Proposition 2.2. *With the above notation, let $Q_i \subseteq \mathbb{R}[X_i]$, $i = 1, 2$, be archimedean quadratic modules.*
If an element $p \in u_1^*\mathbb{R}[X_1] + u_2^*\mathbb{R}[X_2]$ is strictly positive on the compact set $u_1^{-1}K(Q_1) \cap u_2^{-1}K(Q_2)$, then $p \in u_1^*Q_1 + u_2^*Q_2$.

The proposition applies to recover Lasserre's Theorem: let $X_1 = \mathbb{R}^{n_1} \times \mathbb{R}^m$, $X_2 = \mathbb{R}^m \times \mathbb{R}^{n_2}$ and $Y = \mathbb{R}^m$, while f_1, f_2 are the corresponding projection maps onto Y.Then one immediately identifies

$$Z = \mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^{n_2}$$

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Generalization to Projective Systems.

Let I be a non-empty set, endowed with a partial order relation $i \leq j$. A *projective system* of algebraic varieties indexed over I consists of a family of varieties (affine in our case) $X_i,\ i \in I$, and morphisms $f_{ij} : X_j \longrightarrow X_i$ defined whenever $i \leq j$, and satisfying the compatibility condition

$$f_{ik} = f_{ij} f_{jk} \text{ if } i \leq j \leq k.$$

The projective limit $V = \text{proj.lim}(X_i, f_{ij})$ is the universal object endowed with morphisms $f_i : X \longrightarrow X_i$ satisfying the compatibility conditions

$$f_i = f_{ij} f_j, \quad i \leq j.$$

For example, the fibre product is the projective limit of two arrows converging to the same target:

$$X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2.$$

Iterating this construction we obtain varieties of the form

$$((X_1 \times_{Y_1} X_2) \times_{Y_2} X_3) \times_{Y_3} \ldots \times_{Y_n} X_{n+1}.$$

The associated ordered sets belong to the following category: Let $\mathcal{R}$ be the class of partially ordered sets $(I, \leq)$ thus inductively constructed:

- $\{\alpha_1, \alpha_2, \beta; \beta \leq \alpha_1, \beta \leq \alpha_2\} \in \mathcal{R}$;
- $(I, \leq) \in \mathcal{R} \;\Rightarrow I \cup \{\alpha, \beta\} \in \mathcal{R}$;

$\exists\,!\, i(\beta) \in I,\; i(\beta) \geq \beta, \;\; \alpha \geq \beta$

We do not exclude in the second axiom $i(\beta) = \beta$, but we ask α to be an external element of I. For example, the graph

$$* \longrightarrow * \longleftarrow * \longrightarrow * \longleftarrow \dots \longrightarrow *$$

is another example of an ordered set belonging to $\mathcal{R}$.
The relevance to **(RIP)** is summarized in the following obvious observation.

Proposition 2.3. *Let* $\{I_1, ..., I_k\}$ *be a covering of the set of indices* $I = \{1, 2, ..., d\}$ *satisfying the* **(RIP)**. *Then the partially ordered set underlying the projective system of affine spaces and projection maps*

$$\mathbb{R}^{I_{k(j)}} \longrightarrow \mathbb{R}^{I_{k(j)} \cap I_j} \longleftarrow \mathbb{R}^{I_j}, \;\; 2 \leq j \leq k,$$

belongs to the class $\mathcal{R}$.

A repeated use of the Striktpositivstellensatz for fibre products yields:

Theorem 2.4. *Let (X_i, f_{ij}) be a finite projective system of real affine varieties, with the ordered index set belonging to the class $\mathcal{R}$. Let $Q_i \subseteq \mathbb{R}[X_i]$ be archimedean quadratic modules, subject to the coherence condition*

$$f_{ij}^* Q_i \subseteq Q_j \text{ for } i \leq j \, .$$

Let

$$p \in \sum_i f_i^* \mathbb{R}[X_i]$$

be positive on the compact set

$$\mathbf{K} := \cap_{i \in I} f_i^{-1} K(Q_i) \, .$$

Then $p \in \sum_i f_i^* Q_i$.

The coherence condition $(f_{ij})^*Q_i \subseteq Q_j$ implies $f_{ij}K(Q_j) \subseteq K(Q_i)$, that is we can actually work with the projective system $(K(Q_i), f_{ij})$ of compact spaces.

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Proof of the Main Lemma

Let V be a real vector space and let C be a convex cone in V. We say that an element ξ belongs to the *algebraic interior*, in short $\xi \in \text{alg.int}C$, if for every $f \in V$ there exists a positive constant λ such that $\xi + \lambda f \in C$. We recall the following separation lemma (Eidelheit, Kakutani, Krein):

Lemma 2.5. *Let $C \subset V$ be a convex cone in a real vector space V. Assume that $\xi \in \text{alg.int}C$ and that $g \notin C$. Then there exists a linear functional $L \in V'$, such that*

$$L(g) \leq 0 \leq L(c), \quad c \in C; \quad L(\xi) = 1.$$

We need **Bourbaki**:

Lemma 2.6. *Let* $Z = X_1 \times_Y X_2$ *be a (reduced) fibre product of affine, real algebraic varieties, with structural maps* $f_i : X_i \longrightarrow Y$, $u_i : Z \longrightarrow X_i$, $i = 1, 2$. *Let* μ_i *be probability measures on* X_i, $i = 1, 2$, *respectively, satisfying* $(f_1)_*\mu_1 = (f_2)_*\mu_2$. *If the restricted maps* $f_i : \mathrm{supp}\mu_i \longrightarrow Y$, *are proper, then there exists a probability measure* μ *on* Z *satisfying* $(u_i)_*\mu = \mu_i$, $i = 1, 2$, *and with*

$$\mathrm{supp}\mu \subseteq [u_1^{-1}\mathrm{supp}\mu_1] \cap [u_2^{-1}\mathrm{supp}\mu_2].$$

Let $C \subset \mathbb{R}[X]$ be a convex cone, and let V be the linear span of C in $\mathbb{R}[X]$. We say that C satisfies **(SMP)** if every linear functional $L \in V'$ which is nonnegative on C is represented by a positive Borel measure supported on $K(C)$.

Lemma 2.7. *Let* $X_1 \times_Y X_2$ *be the (reduced) fibre product of affine, real algebraic varieties, with structural maps* $f_i : X_i \longrightarrow Y,\ u_i : Z \longrightarrow X_i,\ i = 1, 2$. *Let* $C_i \subseteq \mathbb{R}[X_i]$ *be convex cones with the (SMP) and* $1 \in \text{alg.int} C_i$, *with respect to the linear subspace* V_i *generated by* $C_i,\ i = 1, 2$, *respectively, and such that the maps* $f_i : K(C_i) \longrightarrow Y$ *are proper. Assume that*

$$p \in u_1^* V_1 + u_2^* V_2$$

is positive on

$$u_1^{-1} K(C_1) \cap u_2^{-1} K(C_2) .$$

Then

$$p \in u_1^* C_1 + u_2^* C_2 .$$

Note: The conditions in the statement are met if $C_i \subseteq \mathbb{R}[X_i]$ are archimedean quadratic modules. Indeed, in this case $1 \in \text{alg.int}C_i$ and $K(C_i)$ are compact sets.

3. Proof

Assume that $p \notin u_1^* C_1 + u_2^* C_2$.
Since $1 \in \text{alg.int}[u_1^* C_1 + u_2^* C_2]$,
the separation lemma yields a linear functional $L \in [u_1^* V_1 + u_2^* V_2]'$, non-negative on $u_1^* C_1 + u_2^* C_2$ and satisfying

$$L(p) \leq 0 < L(1) = 1\,.$$

Let $L_i(f) = L(u_i^* f),\ f \in V_i,\ i = 1, 2$.
By assumption, L_i is represented by a probability measure μ_i, supported by the positivity set $K(C_i)$. Moreover

$$(f_1)_* \mu_1 = (f_2)_* \mu_2$$

(because by definition: $\int g\ d(f_1)_* \mu_1 = L(u_1^* f_1^* g) = L(u_2^* f_2^* g) = \int g\, d(f_2)_* \mu_2$).

Since the restricted maps

$$f_i : K(C_i) \longrightarrow Y$$

are proper, **Bourbaki** yields a positive measure μ on

$$S = u_1^{-1}K(C_1) \cap u_2^{-1}K(C_2)$$

which represents the functional L. Consequently,

$$L(p) = \int_S p d\mu > 0,$$

a contradiction.

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