MTNS Conference, Polynomial Inequalities and Applications. Kyoto, July 24-28 2006.

July 20, 2006

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[1]Partially supported by the Natural Sciences and Engineering Research Council of Canada.

Positive Polynomials and Invariant Theory.

As all roads lead to Rome so I find in my own case at least that all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the Theory of Invariants.

– J. J. Sylvester 1854

The slides of this talk are available at:
http://math.usask.ca/~skuhlman/slidesjp.pdf

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Positive Polynomials, Sums of Squares and the multi-dimensional Moment Problem.

In algebraic geometry, we consider ideals of the polynomial ring and algebraic varieties in affine space. In semi-algebraic geometry, we consider preorderings of the polynomial ring and semialgebraic sets in affine space.

• Let $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ be the ring of polynomials in n variables and real coefficients.

• A subset $T \subseteq \mathbb{R}[X]$ is a **quadratic preordering** if $f^2 \in T$, $\forall f \in \mathbb{R}[X]$ and T is closed under addition and multiplication. The smallest preordering of $\mathbb{R}[X]$ is the set of **sums of squares** of $\mathbb{R}[X]$, denoted by $\sum \mathbb{R}[X]^2$.

• Given a subset S of $\mathbb{R}[X]$, there is a smallest preordering T_S containing S; the **preordering generated by** S:

$$T_S = \{ \sum_{e \in \{0,1\}^r} \sigma_e f^e : r \geq 0, \sigma_e \in \sum \mathbb{R}[X]^2, f_1, \cdots, f_r \in S\}$$

where $f^e := f_1^{e_1} \cdots f_r^{e_r}$, if $e = (e_1, \cdots, e_r)$.

• Let $S = \{f_1, \cdots, f_s\} \subset \mathbb{R}[X]$, S defines a **basic closed semialgebraic** subset of $\mathbb{R}^n$:

$$K = K_S = \{x \in \mathbb{R}^n \ : \ f_1(x) \geq 0, \ldots, f_s(x) \geq 0\}.$$

Hilbert's 17th Problem and Stengle's Positivstellensatz are concerned with the issue of representation of positive semi-definite polynomials.

- We consider polynomials that are **positive semi-definite** on K_S:

$$\mathrm{Psd}(K_S) := \{f \in \mathbb{R}[X] : f(x) \geq 0 \text{ for all } x \in K_S\}$$

- $\mathrm{Psd}(K_S)$ is a preordering in $\mathbb{R}[X]$ containing T_S.
- *Question:* Is it true that

$$\mathrm{Psd}(K_S) = T_S\,?$$

- Say that T_S is **saturated** if $\mathrm{Psd}(K_S) = T_S$.

Examples:

• $n = 1$, $S = \emptyset$, $K_S = \mathbb{R}$, $T_S = \Sigma\,\mathbb{R}[X]^2$. It is straightforward to show that a positive semi-definite polynomial on $\mathbb{R}$ is a sum of squares of two polynomials; so in this case, the answer to the above question is yes.

• $n = 1$, $S = \{(1 - X^2)^3\}$, $K_S = [-1, 1]$. Consider $f(X) = (1 - X^2)$, $f \geq 0$ on K_S. An elementary argument shows that $f \notin T_S$; so T_S is not saturated.

• $n = 1$, $S = \{X^3\}$, $K_S = [0, \infty)$. Consider $f(X) = X$, $f \geq 0$ on K_S. An elementary argument shows that $f \notin T_S$; so T_S is not saturated.

• *We shall return to these examples, and give a general criterion for saturated preorderings associated to semi-algebraic subsets of the real line.*

• $n = 2$, $S = \emptyset$, $K_S = \mathbb{R}^2$, $T_S = \Sigma\,\mathbb{R}[X]^2$. Hilbert knew that there exists a polynomial of degree 6 which is positive semi-definite on the real plane, but not a sum of squares. The first explicit example was given by Motzkin:

$$m(X_1, X_2) := X_1^4 X_2^2 + X_2^4 X_1^2 - 3X_1^2 X_2^2 + 1$$

• $n \geq 3$: Scheiderer [9] shows that if $\dim(K_S) \geq 3$, then there exists a polynomial $p(X) \in \mathbb{R}[X]$ such that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$ but $p \notin T_S$ (so T_S cannot be saturated).

It follows from Scheiderer's result that the preordering Psd(K_S) is seldom finitely generated. Another attempt to approximate Psd(K_S) by the finitely generated preordering T_S is related to the multi-dimensional moment problem.

• The general Moment Problem is the following: Given a linear functional $L \neq 0$ on $\mathbb{R}[X]$ and a closed subset K of $\mathbb{R}^n$, when can one find a positive Borel measure μ on K such that for all $f \in \mathbb{R}[X]$

$$L(f) = \int_K f d\mu \,?$$

• Say L is **represented by a measure** μ on K if $\forall f \in \mathbb{R}[X]$

$$L(f) = \int_K f d\mu \,.$$

• The following result is due to Haviland.

Theorem 0.1 *Given a linear functional $L \neq 0$ on $\mathbb{R}[X]$ and a closed subset K of $\mathbb{R}^n$, L is represented by a measure μ on K if and only if $L(Psd(K)) \geq 0$.*

• Since $\mathrm{Psd}(K)$ is in general not finitely generated , we are interested in approximating it by T_S.

• We work with the following corresponding preordering:

$\overline{T_S} := \{f\,;\, L(f) \geq 0 \text{ for all } L \neq 0 \text{ such that } L(T_S) \geq 0\}\,.$

• $\overline{T_S}$ is the **closure** of T_S in $\mathbb{R}[X]$ (for the finest locally convex topology on $\mathbb{R}[X]$).

• We have the inclusions

$$T_S \subseteq \overline{T_S} \subseteq \mathrm{Psd}(K_S)\,.$$

• We say that S **solves the moment problem** if

$$\overline{T_S} = \mathrm{Psd}(K_S)$$

that is, if T_S is dense in $\mathrm{Psd}(K_S)$.

• Given a basic closed semialgebraic set K, we say that the **moment problem is finitely solvable** for K if a finite description S of K can be found such that T_S is dense in $\mathrm{Psd}(K_S)$ (Say (**SMP)** holds).

Remarks:

• By Theorem 0.1, we see that (**SMP**) holds if and only if every $L \neq 0$ which satisfies $L(T_S) \geq 0$ is represented by a positive Borel measure on $K = K_S$.

• If T_S is closed (i.e. $T_S = \overline{T_S}$) then S solves the moment problem if and only if T_S is saturated.

• In [11] Schmüdgen shows that if K_S is compact, then T_S is dense in $\mathrm{Psd}(K_S)$ (i.e. S solves the moment problem).

• In [4], [5] and [6] Schmüdgen's result is extended to cover many non-compact examples.

As an illustration, we discuss (as promised) in the next slide saturated preorderings and solvability of the moment problem for subsets of the real line.

The one-dimensional Moment Problem.

We state [5, Theorem 2.2]. We need to define some notions.

If $K \subseteq \mathbb{R}$ is a non-empty closed semi-algebraic set. Then K is a finite union of intervals. It is easily verified that $K = K_{\mathcal{N}}$, for $\mathcal{N}$ the set of polynomials defined as follows:

•If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $X - a \in \mathcal{N}$.

•If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - X \in \mathcal{N}$.

•If $a, b \in K$, $(a, b) \cap K = \emptyset$, then $(X - a)(X - b) \in \mathcal{N}$.

• $\mathcal{N}$ has no other elements except these.

We call $\mathcal{N}$ **the natural set of generators** for K.

Examples:

• $n = 1$, $K = \mathbb{R}$, $\mathcal{N} = \emptyset$

• $n = 1$, $K = [-1, 1]$, $\mathcal{N} = \{1 + X, 1 - X\}$.

• $n = 1$, $K = [0, \infty)$, $\mathcal{N} = \{X\}$.

The theorem on the next slide ([5, Theorem 2.2]) shows that the one-dimensional moment problem for non-compact subsets of the real line is always solvable. This generalizes several well-known results (Hamburger, Stieljes, Svecov, Hausdorff, etc...). Combined with Schmüdgen's result, we see that the one-dimensional moment problem for subsets of the real line is always solvable.

Theorem 0.2 *Assume that $K = K_S \subseteq \mathbb{R}$ is not compact. Then T_S is closed. (Therefore S solves the moment problem if and only if T_S is saturated). Moreover, T_S is saturated if and only S contains the natural set of generators of K (up to scalings by positive reals).*

For the compact case, we also have a criterion. Assume that K_S has no isolated points :

Theorem 0.3 *Let K_S be compact, $S = \{g_1, \cdots, g_s\}$. Then T_S is saturated if and only if, for each endpoint $a \in K_S$, there exists $i \in \{1, \cdots, s\}$ such that $x - a$ divides g_i but $(x-a)^2$ does not.*

• We now want to extend Schmüdgen's result in another direction. The idea is to fix a distinguished subset $B \subset \mathbb{R}[X]$ and to attempt the various approximations *only for polynomials in B*. That is, we want to study the inclusions

$$T_S \cap B \subseteq \overline{T_S} \cap B \subseteq \mathrm{Psd}(K_S) \cap B \, .$$

Representation of positive semi-definite invariant polynomials.

• Here, we shall embark in a particularly privileged situation when B **is the subring of invariant polynomials** with respect to some action of a group on the polynomial ring $\mathbb{R}[X]$.

• We fix a group G together with

$$\phi : G \to \mathrm{GL}_n(\mathbb{R})$$

a linear representation. We let G act on $\mathbb{R}^n$.

• We define the corresponding action of G on the polynomial ring $\mathbb{R}[X]$: given $p(X) \in \mathbb{R}[X]$, define $p^g(X) := p(\phi(g)X)$.

• Recall that $p(X) \in \mathbb{R}[X]$ is G-invariant if for all $g \in G$: $p^g(X) = p(X)$.

Remarks:

• If $p(X) \in \Sigma\mathbb{R}[\ \underline{X}\ ]^2$ then for all $g \in G$, $p^g(X) \in \Sigma\mathbb{R}[\ \underline{X}\ ]^2$; so $\Sigma\mathbb{R}[\ \underline{X}\ ]^2$ is (setwise) G-invariant.

• If $K \subset \mathbb{R}^n$ is (setwise) G-invariant and $p(X) \in \mathrm{Psd}(K)$, then for all $g \in G$, $p^g(X) \in \mathrm{Psd}(K)$; so $\mathrm{Psd}(K)$ is (setwise) G-invariant.

• If S is a set of invariant polynomials, then K_S and T_S are (setwise) G-invariant.

• Conversely, if $K \subset \mathbb{R}^n$ is G-invariant, it can be described by a set of invariant polynomials; see [[1]; Cor. 5.4].

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Preorderings of the ring of invariant polynomials.

• Write $\mathbb{R}[\,\underline{X}\,]^G$ for the ring of all G-invariant polynomials.

• We shall always assume that G is a **reductive** group. So G admits a **Reynolds operator**. The Reynolds operator is an $\mathbb{R}$-linear map, which is the identity on $\mathbb{R}[\,\underline{X}\,]^G$, and is a $\mathbb{R}[\,\underline{X}\,]^G$-module homomorphism.

• For such groups, Hilbert's Finiteness Theorem is valid; namely $\mathbb{R}[\,\underline{X}\,]^G$ is a finitely generated $\mathbb{R}$-algebra.

• In this talk, for simplicity, we consider the case when G is a finite group. Here, the Reynolds operator is just the **average map**:

$$* : \mathbb{R}[X] \to \mathbb{R}[\,\underline{X}\,]^G, \quad f \mapsto f^* := \frac{1}{|G|} \sum_{g \in G} f^g.$$

• *We use the Reynolds operator as a tool to describe preorderings of* $\mathbb{R}[\,\underline{X}\,]^G$*:*

• If $A \subseteq \mathbb{R}[\,\underline{X}\,]$ we shall denote by A^* its image in $\mathbb{R}[\,\underline{X}\,]^G$ under the Reynolds operator.

• If $A \subseteq \mathbb{R}[\,\underline{X}\,]$, let us denote $A^G := A \cap \mathbb{R}[\,\underline{X}\,]^G$.

• Observe that if T is any preordering in $\mathbb{R}[\,\underline{X}\,]$, then T^G is a preordering of $\mathbb{R}[\,\underline{X}\,]^G$. What about T^*?

• We note the following important property:

Lemma 0.4 *let $A \subseteq \mathbb{R}[\,\underline{X}\,]$. Assume that A is closed under addition and is (setwise) invariant.*
Then $A^* = A^G$.

• *Example*: the image under the Reynolds operator of $\sum \mathbb{R}[\,\underline{X}\,]^2 \subset \mathbb{R}[\,\underline{X}\,]$ is a preordering $(\sum \mathbb{R}[\,\underline{X}\,]^2)^G$ of $\mathbb{R}[\,\underline{X}\,]^G$ of **invariant sums of squares**.

•*Remark:* In general

$$\Sigma(\mathbb{R}[\,\underline{X}\,]^G)^2 \subseteq (\Sigma\mathbb{R}[\,\underline{X}\,]^2)^G$$

but this inclusion may be proper. Even worse, the preordering $(\Sigma\mathbb{R}[\,\underline{X}\,]^2)^G$ *need not* be finitely generated as a preordering of $\mathbb{R}[\,\underline{X}\,]^G$ ([3]).

• *We denote by S_0 a set of generators of* $(\Sigma\mathbb{R}[\,\underline{X}\,]^2)^G$ *(as a preordering of* $\mathbb{R}[\,\underline{X}\,]^G$*).*

Proposition 0.5 *Let $n = 1$ and $G = \{-1, 1\}$. We claim that $S_0 = \{X^2\}$ generates the preordering $(\sum \mathbb{R}[\underline{X}]^2)^G$ over $\Sigma(\mathbb{R}[\underline{X}]^G)^2$.*

Proof: Indeed if $\sigma \in (\sum \mathbb{R}[\underline{X}]^2)^G$, then

$$\sigma = \sigma^* = \sum_i (\eta_i^2)^* \text{ with } \eta_i(X) \in \mathbb{R}[X] \, .$$

Now $(\eta_i^2)^*(X) = \eta_i^2(X) + \eta_i^2(-X)$, so it suffices to prove the claim for $\eta_i^2(X) + \eta_i^2(-X)$.

By separating terms of even and odd degree, we can write

$$\eta(X) = \mu(X^2) + X\theta(X^2) \, ,$$

with appropriately chosen $\mu(X), \theta(X) \in \mathbb{R}[X]$. Therefore

$$\eta_i^2(X) + \eta_i^2(-X) = (\mu(X^2) + X\theta(X^2))^2 + (\mu(X^2) - X\theta(X^2))^2 = 2\mu(X^2)^2 + 2X^2\theta(X^2)^2$$

which is an element of the preordering of $\mathbb{R}[\underline{X}]^G$ generated by X^2 as required. $\square$

In the next slide, we continue our analysis of the preorderings of $\mathbb{R}[\underline{X}]^G$.

• Let $S = \{f_1, \ldots, f_k\} \subset \mathbb{R}[X]^G$, and $K_S \subset \mathbb{R}^n$ the invariant basic closed semialgebraic set defined by S.

• We are particularly interested in the following three preorderings of $\mathbb{R}[X]^G$, associated to S:

• The preordering of G-invariant psd polynomials $\mathrm{Psd}^G(K_S)$.

• The preordering T_S^G.

• The preordering $T_S^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}}$ of $\mathbb{R}[X]^G$ which is finitely generated by S.

• We have: $T_S^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}} \subseteq T_S^G \subseteq \mathrm{Psd}^G(K_S)$.

• The preordering T_S^G is easy to describe:

Lemma 0.6 *T_S^G is the preordering of $\mathbb{R}[\ \underline{X}\ ]^G$ generated by $(\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G$ and S.*

Proof: Let $h \in T_S^G$. Write

$$h = \sum_{e \in \{0,1\}^s} \sigma_e f^e, \text{ with } \sigma_e \in \sum \mathbb{R}[X]^2$$

for some $\{f_1, \ldots, f_k\} \subseteq S$. Applying the Reynolds operator we get

$$h = h^* = (\sum_{e \in \{0,1\}^s} \sigma_e f^e)^* = \sum_{e \in \{0,1\}^s} \sigma_e^* f^e$$

(since $f_1, \cdots, f_s \in \mathbb{R}[\ \underline{X}\ ]^G$). This is of the required form since $\sigma_e^* \in (\Sigma\,\mathbb{R}[\ \underline{X}\ ]^2)^G$ for each e. $\square$

Semi-Algebraic Geometry in the Orbit Space.

Fix $p_1, \cdots, p_k \in \mathbb{R}[X]$ *generators of* $\mathbb{R}[X]^G$.

• Consider the polynomial map

$\pi : \mathbb{R}^n \to \mathbb{R}^k,\ \underline{a} = (a_1, \cdots, a_n) \mapsto (p_1(\underline{a}) \cdots, p_k(\underline{a}))$.

• By [[1]; Proposition 5.1], the image of an invariant basic closed semi-algebraic set is a basic closed semi-algebraic set. In particular is $\pi(\mathbb{R}^n)$ is basic closed semi-algebraic.

• Let $\mathbb{R}[U]$:= the polynomial ring $\mathbb{R}[U_1, \cdots, U_k]$ in k-variables. Fix a finite description $v_1, \cdots, v_r \in \mathbb{R}[U]$ of $\pi(\mathbb{R}^n)$.

• For the remaining of the talk, we assume that the finite group G is a **generalized reflection** group. In this case, $\mathbb{R}[X]^G$ is generated by $k = n$ algebraically independent elements (see [9]).

• We let

$$\tilde{\pi} : \mathbb{R}[X]^G \to \mathbb{R}[U] = \mathbb{R}[U_1, \cdots, U_n]$$

be the induced $\mathbb{R}$-algebra isomorphism mapping p_i to U_i. We have

$$\tilde{\pi}^{-1}(f)(\underline{a}) = f(p_1(\underline{a}) \cdots, p_k(\underline{a})) \text{ for all } \underline{a} \in \mathbb{R}^n .$$

G–Saturation.

We conclude this talk by a discussion of the notion of G-saturation, which illustrates our "going down" idea.

• We say that T_S^G is saturated, or that T_S **is G-saturated** if every polynomial which is positive semi-definite *and invariant* is represented in the preordering, that is, if

$$T_S^G = \mathrm{Psd}(K_S)^G \, .$$

Proposition 0.7 *If $T_{\tilde{\pi}(S)\cup\tilde{\pi}(S_o)\cup\{v_1,\cdots v_r\}}$ is saturated as a preordering of $\mathbb{R}[U]$, then T_S is G–saturated.*

Let $n = 1$ and G as in example 0.5. Then $\mathbb{R}[X]^G$ is generated by the polynomial $p_1(X) = X^2$ and $\pi(\mathbb{R})$ is the positive half line and is defined by $v_1 = U$. Combining Proposition 0.7 with [5, Theorem 2.2] we obtain the following variant of [5, Theorem 2.2]:

Theorem 0.8 *Let $S \subset \mathbb{R}[\ \underline{X}\ ]^G$. Assume that K_S is non-compact. Assume that: if $(a, b), 0 < a < b$ is a connected component of $\mathbb{R} \setminus K_S$ then S contains (up to a scalar multiple) $(x^2 - a^2)(x^2 - b^2)$, if $(-a, a)$ is a connected component of $\mathbb{R} \setminus K_S$, then S contains (up to a scalar multiple) $x^2 - a^2$. Then T_S is G-saturated,*

• This provides many examples of non-saturated preorderings that are G–saturated.

The End

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