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Salma Kuhlmann [1] ,
Research Center Algebra, Logic and Computation
University of Saskatchewan,
McLean Hall, 106 Wiggins Road,
Saskatoon, SK S7N 5E6, Canada

email: skuhlman@math.usask.ca

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http://math.usask.ca/~skuhlman/slidesim.pdf

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Approximation of positive polynomials by sums of squares.

Introduction.

In algebraic geometry, we consider ideals of the polynomial ring and algebraic varieties in affine space. In semi-algebraic geometry, we consider preorderings of the polynomial ring and semialgebraic sets in affine space.

• Let $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ be the ring of polynomials in n variables and real coefficients.

• A subset $M \subseteq \mathbb{R}[X]$ is a **quadratic module** if $1 \in M$, M is closed under addition and multiplication by squares (i.e. $a^2 f \in M$, $\forall a \in \mathbb{R}[X]$ and $f \in M$).

• A **quadratic preordering** is a quadratic module which is also closed under multiplication.

• The smallest preordering of $\mathbb{R}[X]$ is the set of **sums of squares** of $\mathbb{R}[X]$, denoted by $\sum \mathbb{R}[X]^2$.

• Given a finite subset $S = \{f_1, ..., f_s\}$ of $\mathbb{R}[X]$, the smallest preordering containing S (**preordering finitely generated by** S) is:

$$T_S = \{ \sum_{e \in \{0,1\}^s} \sigma_e f^e : \sigma_e \in \sum \mathbb{R}[X]^2, f_1, \cdots, f_s \in S\}$$

where $f^e := f_1^{e_1} \cdots f_r^{e_s}$, if $e = (e_1, \cdots, e_s)$.

• The smallest module containing S (**module finitely generated by** S) is:

$$M_S = \{\sigma_0 + \sigma_1 f_1 + ... + \sigma_s f_s \; ; \sigma_e \in \sum \mathbb{R}[X]^2 \, .\}$$

• Let $S = \{f_1, \cdots, f_s\} \subset \mathbb{R}[X]$, S defines a **basic closed semialgebraic** subset of $\mathbb{R}^n$:

$$K = K_S = \{x \in \mathbb{R}^n \ : \ f_1(x) \geq 0, \ldots, f_s(x) \geq 0\}$$

• Consider polynomials **positive semi-definite** on K_S:

$$\mathrm{Psd}(K_S) := \{f \in \mathbb{R}[X] : f(x) \geq 0 \text{ for all } x \in K_S\}$$

• $\mathrm{Psd}(K_S)$ is a preordering in $\mathbb{R}[X]$ and $T_S \subseteq \mathrm{Psd}(K_S)$.

Hilbert's 17th Problem and Stengle's Positivstellensatz are concerned with the issue of representation of positive semi-definite polynomials; motivated by the question: when it true that $Psd(K_S) = T_S$?

More generally, we are concerned with the issue of approximating $\mathrm{Psd}(K_S)$ by "smaller" preorderings (modules):

$$T_S^{\dagger} = \{f \;:\; \forall \text{ real } \epsilon > 0\,,\; f + \epsilon \in T_S\}.$$

$$T_S^{\ddagger} = \{f : \exists q \in \mathbb{R}[X] \text{ such that } \forall \text{ real } \epsilon > 0, f+\epsilon q \in T_S\}.$$

$$\overline{T_S} := \{f : L(f) \geq 0, \forall \text{ lin. funct. } L \neq 0 \text{ on } \mathbb{R}[X] \text{ s. t. } L(T_S) \geq 0\}.$$

We have:

$$T_S \subseteq T_S^{\dagger} \subseteq T_S^{\ddagger} \subseteq \overline{T_S} \subseteq \mathrm{Psd}(K_S)\,.$$

We say:

• T_S is **saturated** if $\text{Psd}(K_S) = T_S$.

• $(\dagger)_S$ holds if $T_S^\dagger = \text{Psd}(K_S)$

• $(\ddagger)_S$ holds if $T_S^\ddagger = \text{Psd}(K_S)$

• S **solves the** K_S- **moment problem** if $\overline{T_S} = \text{Psd}(K_S)$. Note that this is equivalent to saying that T_S is *dense* in $\text{Psd}(K_S)$.

Remark 0.1 (i) $\overline{T_S}$ is the **closure** of T_S in $\mathbb{R}[X]$ (for the finest locally convex topology on $\mathbb{R}[X]$).

Denote by P_d the (finite dimensional) vector space consisting of all polynomials in $\mathbb{R}[X]$ of degree $\leq 2d$, and by $T_d = T_S \cap P_d$. The set T_d is obviously a bf cone in P_d, i.e., $T_d + T_d \subseteq T_d$ and $\mathbb{R}^+ T_d \subseteq T_d$. Denote by $\overline{T}_d$ the closure (in the Euclidean topology) of T_d in P_d. Then:

(ii) $T_S^\ddagger = \cup_{d \geq 0} \overline{T}_d$.

(iii) The containments (end of page 4) may be strict. The conjecture that $T_S^\ddagger \neq \overline{T_S}$ was given in [K–M] and recently proved in [N].

(iv) All the above, except for $\text{Psd}(K_S)$ *depend in general on the choice of the description* S *of* $K = K_S$.

PLAN OF THE TALK

In this talk I will give an Atlas of what is known about the various approximations of $Psd(K_S)$ *by those preorderings (or the corresponding module versions) depending on the description* S*, the dimension of the semi-algebraic set* K_S*, intrinsic geometric properties of* K_S *(e.g. compact or unbounded), and special properties of* K_S *(symmetry, sparse representation)*:

(I) Saturation.

(II) The dagger condition.

(III) The double-dagger condition.

(IV) The density condition.

(V) Special situations.

Saturation

In [S1] Scheiderer showed:

Theorem 0.2 *If dim(K_S) $\geq$ 3, then there exists a polynomial $p(X) \in \mathbb{R}[X]$ such that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$ but $p \notin T_S$ (so T_S cannot be saturated).*

Scheiderer's result is intrinsic; under this hypothesis on $K = K_S$, independently of the chosen description S, and whether K_S is compact or unbounded, the preordering T_S cannot be saturated.

In the same paper, he also shows another intrinsic result:

Theorem 0.3 *If $n = 2$ and K_S contains a cone of dimension 2, then there exists a polynomial $p(X) \in \mathbb{R}[X]$ such that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$ but $p \notin T_S$ (so T_S cannot be saturated).*

Low dimensional sets:

This left open the question formulated in [K-M]: what if $K_S \subseteq \mathbb{R}^2$ does not contain a cone of dimension 2? Are there compact/noncompact examples of such K_S for which T_S is saturated?

Recently, Scheiderer developed in a series of papers [S2], [S3], [S4] several *local global principles* to determine when a polynomial $f \geq 0$ on K_S belongs to the *quadratic module* M_S. His results generalize both Schmüdgen's and Putinar's Striktpositivstellensätze. With these tools, he was able to produce the example that we were looking for:

Example 0.4 The modules generated by:
$S_1 = \{1+x, 1-x, 1+y, 1-y\}$ (compact K_S) and
$S_2 = \{x, 1-x, y, 1-xy\}$ (noncompact K_S)
are saturated.

In [K-M-S], we studied saturated preorderings (modules) for subsets of the real line. We discuss the case $n = 1$ *in the next slides.*

To state [K-M-S; Theorem 2.2]. We need to define some notions.

If $K \subseteq \mathbb{R}$ is a non-empty closed semi-algebraic set. Then K is a finite union of intervals. It is easily verified that $K = K_{\mathcal{N}}$, for $\mathcal{N}$ the set of polynomials defined as follows:
•If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $X - a \in \mathcal{N}$.
•If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - X \in \mathcal{N}$.
•If $a, b \in K$, $(a, b) \cap K = \emptyset$, then $(X - a)(X - b) \in \mathcal{N}$.
• $\mathcal{N}$ has no other elements except these.

We call $\mathcal{N}$ **the natural set of generators** for K.

We first consider the non-compact case:

Theorem 0.5 *Assume that $K = K_S \subseteq \mathbb{R}$ is not compact. Then T_S is saturated if and only S contains the natural set of generators of K (up to scalings by positive reals).*

For the compact case, we also have a criterion. Assume that K_S has no isolated points :

Theorem 0.6 *Let K_S be compact, $S = \{g_1, \cdots, g_s\}$. Then T_S is saturated if and only if, for each endpoint $a \in K_S$, there exists $i \in \{1, \cdots, s\}$ such that $x - a$ divides g_i but $(x - a)^2$ does not.*

What about the module version?

In [K-M-S] we asked whether $M_S = T_S$ if $K_S \subseteq \mathbb{R}$ is compact. Scheiderer provided a positive answer using his local-global criteria. In [F] another elementary proof of this fact is given. Thus the above theorem is a criterion for the quadratic module M_S to be saturated.

The dagger condition

In [Sc1] Schmüdgen proved the following intrinsic result:

Theorem 0.7 *If K_S is compact, then $(\dagger)_S$ holds for T_S.*

A quadratic module M is **archimedean** if for all $f \in \mathbb{R}[X]$, there exists an integer $n \geq 1$ such that $n - f$ and $n + f \in M$. Putinar proved the following result:

Theorem 0.8 *If M_S is archimedean, then $(\dagger)_S$ holds for M_S.*

Remark 0.9 (i) If T_S (or M_S) is archimedean then K_S is compact.
(ii) Wörman showed that if K_S is compact then T_S is archimedean (providing a proof of Schmüdgen's Theorem via the Kadison-Dubois Theorem).
(iii) If K_S is compact, M_S need *not* be archimedean.

What if K_S is not compact?

Apart from the non-compact examples of dimension ≤ 2 presented in the previous section, no non-compact examples in dimension ≥ 3 are known. This motivated considering $(\ddagger)$ instead, as we shall see in the next section.

The doubledagger condition

Non-compact examples by dimension extension.

In [K-M] we construct a large number of non-compact examples where (‡) holds.

Let $S \subseteq \mathbb{R}[X]$ finite and set $p = 1 + \Sigma_{i=1}^{n} X_i^2$.

Denote by $\mathbb{R}[X, Y]$ the polynomial ring in $n+1$ variables $X = X_1, \ldots, X_n, Y$ and consider the finite set

$S' = S \cup \{1 - pY, -(1 - pY)\}$ in $\mathbb{R}[X, Y]$.

Then $K_{S'}$ consists of those points on the hypersurface

$$H = \{(x, y) \in \mathbb{R}^{n+1} \mid p(x)y = 1\}$$

in $\mathbb{R}^{n+1}$ which map to K_S under the projection $(x, y) \mapsto x$.

Theorem 0.10 $(\ddagger)'_S$ *holds.*

Cylinders with compact base.

We continue to denote by $\mathbb{R}[X,Y]$ the polynomial ring in $n+1$ variables $X_1,\dots,X_n,Y$.

Consider a subset $S=\{g_1,\dots,g_s\}$ of $\mathbb{R}[X,Y]$ where the polynomials $g_1,\dots,g_s$ involve only the variables $X_1,\dots,X_n$.

So K_S has the form $K\times\mathbb{R}$, $K\subseteq\mathbb{R}^n$. We further assume that K is compact.

We describe this situation by saying that K_S is a cylinder with compact cross-section. In [K-M] we prove:

Theorem 0.11 *If K_S is a cylinder with compact cross-section, then $(\ddagger)_S$ holds.*

More precisely, let $f\in\mathbb{R}[X,Y]$ is such that $f\geq 0$ on K_S. Let $d\geq 1$ so that the degree of f as a polynomial in Y is $\leq 2d$. Set

$$q(Y):=3+Y+3Y^2+Y^3+\ldots+3Y^{2d}\,.$$

Then for all $\epsilon>0$, $f+\epsilon q(Y)\in T_S$.

Closed Polyhedra.

In [K-M-S] we develop a "fiber criterion" for (‡) to hold on *subsets* of cylinders. In particular, we get an application to generalized polyhedra.

Assume that K_S is the basic closed semi-algebraic set in $\mathbb{R}^m$, $m \geq 1$, defined by $S = \{\ell_1, \dots, \ell_s\}$, where $\ell_1, \dots, \ell_s$ are linear, so K_S is a **closed polyhedron**. If K_S is compact then, by [J–P], (†) holds for M_S.

What if K_S is not compact?

If K_S contains a cone of dimension 2 then, by [K–M] (‡) fails for T_S.

In [K–M] we asked whether (‡) holds in the remaining case, i.e., when K_S is not compact and does not contain a cone of dimension 2.

In [K–M–S] we settle this question completely:

Theorem 0.12 *Let P be a closed polyhedron in $\mathbb{R}^m$ defined by a finite set S of linear polynomials.*
(i) If P is compact then (†) holds for M_S.
(ii) If P is not compact but does not contain a 2-dimensional cone then (‡) *holds for M_S.*
(iii) If P contains a 2-dimensional cone then (‡) *fails for T_S.*

The density condition.

All the previous examples, compact or not, satisfying one of the previous conditions considered, satisfy the density condition. In [Sc2], Schmüdgen gives other methods to produce examples where the density condition holds. In [K-M] we gave an intrinsic condition for the density condition *to fail*:

Theorem 0.13 *The density condition fails whenever* $n \geq 2$ *and* K_S *contains a cone of dimension 2.*

In [P–S] a stronger intrinsic condition is given (if K_S contains a "nasty curve" then the density condition fails). The following example is particularly interesting:

Example 0.14 Consider

$$K := \{(x, y) \in \mathbb{R}^2 \colon -1 \leq (x^2 - 1)(y^2 - 1) \leq 0\}$$

in the plane $\mathbb{R}^2 = V(\mathbb{R})$ (see figure 1).

Arguing using the Powers-Scheiderer condition, one shows now that K-moment problem is not finitely solvable.

Note however that the given set is very special; it displays interesting symmetries. This motivates the next section.

Special Situations.

Invariant Sets.

We can extend the results of the previous sections in another direction.

The idea is to fix a distinguished subset $B \subset \mathbb{R}[X]$ and to attempt the various approximations *only for polynomials in B*. That is, we want to study the inclusions

$$T_S \cap B \subseteq \overline{T_S} \cap B \subseteq \mathrm{Psd}(K_S) \cap B\ .$$

In [C–K–S], we investigated the particularly privileged situation when B **is the subring of invariant polynomials** with respect to some action of a group on the polynomial ring $\mathbb{R}[X]$.

Let us revisit the last example of the last section:

Example 0.15 K is G-invariant, where $G = D_4$ the dihedral group of order eight acting on $\mathbb{R}^2$ in the natural way (as the symmetry group of a square centered at the origin).

The ring of invariants is $\mathbb{R}[x, y]^G = \mathbb{R}[u, v]$ with
$$u = x^2 + y^2, \quad v = x^2y^2,$$
and the orbit variety $W = \mathbb{R}^2//G$ is itself an affine plane.

The image of $\pi: V(\mathbb{R}) \to W(\mathbb{R})$ is
$$Z = \pi(\mathbb{R}^2) = \{(u, v) \in \mathbb{R}^2: u \geq 0,\ v \geq 0,\ u^2 \geq 4v\}.$$

Since $(x^2 - 1)(y^2 - 1) = v - u + 1$, we have
$$\pi(K) = \{(u, v) \in \mathbb{R}^2: v \geq 0,\ 1 \leq u - v \leq 2\}.$$
This is a (half-) strip in the (u, v)-plane (see figure 2):

The moment problem for $\pi(K)$ is solved by the preordering N in $\mathbb{R}[u, v] = W(\mathbb{R})$ generated by v, $u - v - 1$ and $2 - u + v$ (by [K-M-S]). This means that the G-invariant K-moment problem is solvable (i.e. *invariant* linear functional non-negative on the finitely generated preordering is represented by an invariant measure).

Positive polynomials on fibre products.

Throughout this section, a real algebraic, affine variety $V \subseteq \mathbb{R}^d$ is the common zero set of a finite set of polynomials.

The algebra of regular functions on V (the coordinate ring of V) is $\mathbb{R}[V] = \mathbb{R}[X]/I(V)$, where $I(V)$ is the radical ideal associated to V.

The non-negativity set of a subset $S \subset \mathbb{R}[V]$ is

$$K(S) = \{x \in V; f(x) \geq 0, \quad f \in S\}.$$

Let I be a non-empty set, endowed with a partial order relation $i \leq j$. A *projective system* of algebraic varieties indexed over I consists of a family of varieties (affine in our case) $V_i,\ i \in I$, and morphisms $f_{ij} : V_j \longrightarrow V_i$ defined whenever $i \leq j$, and satisfying the compatibility condition

$$f_{ik} = f_{ij} f_{jk} \text{ if } i \leq j \leq k.$$

The topological projective limit $V = \text{proj.lim}(V_i, f_{ij})$ is the universal object endowed with morphisms

$$f_i : V \longrightarrow V_i$$

satisfying the compatibility conditions

$$f_i = f_{ij} f_j, \quad i \leq j.$$

A *directed projective system* carries the additional assumption on the index set that for every pair $i, j \in I$ there exists $k \in I$ satisfying $i \leq k$ and $j \leq k$.

A finite partially ordered set $I = \{i_0, ..., i_n\}$ is a *rooted tree* if the order structure is generated by the inequalities

$$i_1 \geq i_0 \text{ and for every } k > 1, i_k \geq i_{j(k)} \text{ with } j(k) < k.$$

In [K–P] We are concerned with finite projective systems of algebraic varieties. The main result is the following:

Theorem 0.16 *Let* (V_i, f_{ij}) *be a finite projective system of real affine varieties, indexed over a rooted tree. Let* $Q_i \subset \mathbb{R}[V_i]$ *be archimedean quadratic modules, subject to the coherence condition* $f_{ij}^* Q_i \subseteq Q_j$*. Let* $p \in \Sigma_i f_i^* \mathbb{R}[V_i]$ *be an element which is positive on the set* $\cap_{i \in I} f_i^{-1} K(Q_i)$*. Then* $p \in \Sigma_i f_i^* Q_i$*.*

We also consider fibre products of affine real varieties: Let $Z = X_1 \times_Y X_2$ be the fibre product of affine real varieties. Specifically

$$f_i; X_i \longrightarrow Y, \quad i = 1, 2,$$

are given morphisms and

$$Z = \{(x_1, x_2) \in X_1 \times X_2;\ f_1(x_1) = f_2(x_2)\}.$$

This is still an algebraic variety, with the ring of regular functions

$$\mathbb{R}[X_1 \times_Y X_2] = \mathbb{R}[X_1] \otimes_{\mathbb{R}[Y]} \mathbb{R}[X_2].$$

Denote by $u_i : Z \longrightarrow X_i,\ i = 1, 2,$ the projection maps, so that: $f_1 u_1 = f_2 u_2$.

Proposition 0.17 *With the above notation, let* $Q_i \subseteq \mathbb{R}[X_i]$, $i = 1, 2$, *be archimedean quadratic modules. If an element* $p \in u_1^*\mathbb{R}[X_1] + u_2^*\mathbb{R}[X_2]$ *is strictly positive on the set* $u_1^{-1}K(Q_1) \cap u_2^{-1}K(Q_2)$, *then* $p \in u_1^*Q_1 + u_2^*Q_2$.

The proposition applies to the case of fibre products of affine spaces to recover a result of [L].

Specifically, let $X_1 = \mathbb{R}^{n_1} \times \mathbb{R}^m$, $X_2 = \mathbb{R}^m \times \ R^{n_2}$ and $Y = \mathbb{R}^m$, while f_1, f_2 are the corresponding projection maps onto Y. Denote by x_1, y, x_2 the corresponding tuples of variables. Then one immediately identifies

$$Z = \mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^{n_2}$$

and the proposition yields:

Corollary 0.18 *Let* $Q_{x_1,y}, Q_{y,x_2}$ *be archimedean quadratic modules in the respective sets of variables. Let* $\Pi := (K(Q_{x_1,y}) \times \mathbb{R}^{n_2}) \cap (\mathbb{R}^{n_1} \times K(Q_{y,x_2})) \subseteq Z$. *If a polynomial* $p(x_1, y, x_2) = p_1(x_1, y) + p_2(y, x_2)$ *is positive on* Π, *then* $p \in Q_{x_1,y} + Q_{y,x_2}$.

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The End

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