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Moment problem for symmetric algebras of locally convex spaces

July 14, 2016

The moment problem

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The multidimensional moment problem

When is L representable as an integral with respect to a positive Radon measure μ on $\mathbb{R}^n$, i.e.

$$L(f) = \int f \, d\mu, \quad \forall f \in \mathbb{R}[\underline{X}],$$

and what is the support of μ?

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$$\begin{aligned} \hat{a} : X(A) &\to \mathbb{R} \\ \alpha &\mapsto \alpha(a) \end{aligned}$$

$X(A)$ is given the weakest topology such that the functions $\hat{a}$, $a \in A$ are continuous.

Definitions and Notations

- Fix $d \geq 1$. A subset $M \subseteq A$ is a 2d-power module if:
  - M is a cone:
  $$0, 1 \in M, \quad M + M \subseteq M \text{ and } [0, \infty) \cdot M \subseteq M.$$
  - $\forall f \in A \quad f^{2d} M \subseteq M.$

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- $\sum A^{2d}$ is the unique smallest $2d$-power module of A.
- $\sum A^{2d}$ is closed under multiplication, so $\sum A^{2d}$ is also the unique smallest $2d$-power preordering of A.
- A linear functional $L : A \to \mathbb{R}$ is said to be positive if $L(\sum A^{2d}) \subseteq [0, \infty)$.

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We note that the moment problem for $\mathbb{R}[\underline{x}]$ is a special case. Indeed, ring homomorphisms from $\mathbb{R}[\underline{x}]$ to $\mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^n$ and $X(\mathbb{R}[\underline{x}])$ is identified (as a topological space) with $\mathbb{R}^n$.

Seminorms on $\mathbb{R}$- Vector Spaces and $\mathbb{R}$-Algebras

A map $\rho : A \longrightarrow [0, \infty)$ is called a seminorm if

1. $\forall a \in A \, \forall r \in \mathbb{R} \quad \rho(ra) = |r|\rho(a)$,
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is a compact Hausdorff space.

Locally Multiplicatively Convex Topologies

Let $\mathcal{F}$ be a family of submultiplicative seminorms on A. The family $\mathcal{F}$ induces a locally convex topology $\tau_{\mathcal{F}}$ on A such that $(A, \tau_{\mathcal{F}})$ is a topological algebra.

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Wlog we assume that the family $\mathcal{S}$ is directed, i.e., $\forall\, \rho_1, \rho_2 \in \mathcal{S},\ \exists\, \rho \in \mathcal{S}$ such that $\rho \succeq \max\{\rho_1, \rho_2\}$.

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With this assumption, the open balls

$$U_r(\rho) := \{v \in V : \rho(v) < r\},\ \rho \in \mathcal{S},\ r > 0$$

form a basis of neighbourhoods of zero (not just a subbasis).

Proposition

Suppose τ is a locally convex topology generated by a directed family $\mathcal{F}$ of seminorms and L is a τ-continuous linear functional. Then there exists $\rho \in \mathcal{F}$ such that L is ρ-continuous (and conversely, of course).

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Corollary

If $\mathcal{F}$ is directed then $\mathfrak{sp}(\tau_{\mathcal{F}}) = \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}(\rho)$.

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For each submultiplicative seminorm ρ on A and each integer $d \geq 1$ there is a natural one-to-one correspondence $L \leftrightarrow \mu$ given by $L(a) = \int \hat{a} d\mu \; \forall \, a \in A$ between ρ-continuous, positive linear functionals $L : A \to \mathbb{R}$ and positive Radon measures μ on $X(A)$ supported by $\mathfrak{sp}(\rho)$.

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The theorem extends to general LMC topologies. By the Proposition above, the unique Radon measure μ corresponding to a τ-continuous, positive linear functional $L : A \to \mathbb{R}$ is supported by the compact set $\mathfrak{sp}(\rho)$ for some $\rho \in \mathcal{F}$.

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The results apply in some interesting cases. We now study the main application in [GIKM, submitted 2015].

Background: The Symmetric Algebra

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Let V be an $\mathbb{R}$-vector space. We denote by $S(V)$ the symmetric algebra of V, i.e., the tensor algebra $T(V)$ factored by the ideal generated by the elements $v \otimes w - w \otimes v$, $v, w \in V$.

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If we fix a basis x_i, $i \in \Omega$ of V, then $S(V)$ is identified with the polynomial ring $\mathbb{R}[x_i : i \in \Omega]$, i.e., the free $\mathbb{R}$-algebra in commuting variables x_i, $i \in \Omega$.

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Here, $f_{ij} \in V$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$ and $n \geq 1$. Note that $S(V)_0 = \mathbb{R}$ and $S(V)_1 = V$.

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$$(\rho_1 \otimes \rho_2)(f) := \inf\{\sum_{i=1}^{n} \rho_1(f_{i1})\rho_2(f_{i2}) : f = \sum_{i=1}^{n} f_{i1} \otimes f_{i2},\ f_{ij} \in V_j,\ n \geq 1\}.$$

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If all the (V_i, ρ_i) are equal, say $(V_i, \rho_i) = (V, \rho)$, $i = 1, \ldots, k$, the associated tensor seminorm $\rho_1 \otimes \cdots \otimes \rho_k$ on $V^{\otimes k}$ is denoted $\rho^{\otimes k}$.

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Proposition

Let $f \in S(V)_i$, $g \in S(V)_j$, then $\overline{\rho}_{i+j}(fg) \leq \overline{\rho}_i(f)\overline{\rho}_j(g)$.

We now extend ρ to a $\overline{\rho}$ on $S(V)$ as follows: For $f = f_0 + \cdots + f_\ell$, $f_k \in S(V)_k$, $k = 0, \ldots, \ell$, define

$$\overline{\rho}(f) := \sum_{k=0}^{\ell} \overline{\rho}_k(f_k).$$

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We are now in a position to apply [GKM 2014]. We still need to determine explicitly the character space and the Gelfand spectrum.

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Suppose now that A is an $\mathbb{R}$-algebra equipped with σ and $\pi : V \to A$ is $\mathbb{R}$-linear and continuous w.r.t. ρ and σ, i.e. there is $C > 0$ such that $\sigma(\pi(f)) \leq C\rho(f)\ \forall f \in V$.

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Suppose now that A is an $\mathbb{R}$-algebra equipped with σ and $\pi : V \to A$ is $\mathbb{R}$-linear and continuous w.r.t.ρ and σ, i.e.there is $C > 0$ such that $\sigma(\pi(f)) \leq C\rho(f)\ \forall f \in V$. It follows that: $\sigma(\overline{\pi}(f)) \leq C^k \overline{\rho}_k(f)\ \forall f \in S(V)_k$.

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Proposition

If $\pi : (V, \rho) \to (A, \sigma)$ has operator norm ≤ 1, then the induced algebra homomorphism $\overline{\pi} : (S(V), \overline{\rho}) \to (A, \sigma)$ has operator norm $\leq \sigma(1)$.

The Character Space and the The Gelfand Spectrum

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Proposition

$\mathfrak{sp}(\overline{\rho})$ is naturally identified with the closed ball $\overline{B}_1(\rho')$. Here ρ' denotes the operator norm on V^*, i.e.,
$\rho'(v^*) := \inf\{C \in [0, \infty) : |v^*(w)| \leq C\rho(w)\ \forall w \in V\}$.

The Moment Problem

Main Corollary I

For each seminormed $\mathbb{R}$-vector space (V, ρ) and each integer $d \geq 1$ there is a natural one-to-one correspondence $L \leftrightarrow \mu$ given by $L(f) = \int \hat{f} d\mu \; \forall f \in S(V)$ between $\overline{\rho}$-continuous, positive linear functionals $L : S(V) \to \mathbb{R}$ and positive Radon measures μ on V^* supported by $\overline{B}_1(\rho')$.

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Main Corollary II

Let τ be the locally convex topology on an $\mathbb{R}$-vector space V defined by a directed family $\mathcal{S}$ of seminorms on V. For each integer $d \geq 1$ there is a natural one-to-one correspondence $L \leftrightarrow \mu$ given by $L(f) = \int \hat{f} d\mu \; \forall f \in S(V)$ between $\overline{\tau}$-continuous, positive linear functionals $L : S(V) \to \mathbb{R}$ and positive Radon measures μ on V^* supported by $\overline{B}_i(\rho')$ for some $\rho \in \mathcal{S}$ and some integer $i \geq 1$. If μ is supported by $\overline{B}_i(\rho')$ then L is $\overline{i\rho}$-continuous, and conversely.

Thank you