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EMBEDDING ORDERED FIELDS IN FORMAL POWER SERIES FIELDS

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Let F be an ordered field, v the unique finest valuation on F compatible with the ordering (so $v(a) \geq v(b)$ iff $n|b| \geq |a|$ for some integer $n \geq 1$), V the value group of v and κ the residue field of v (so κ is archimedean) [5] [10] [19]. If F' is an ordered extension of F, then the finest valuation on F' compatible with the extended ordering is an extension of v which we denote also by v. We denote by V' and κ' the value group and residue field of the extended v.

$\overline{F}$ denotes the real closure of F. The residue field in this case is $\overline{\kappa}$, the real closure of κ. The value group is $\overline{V}$:= the divisible hull of V. Any power series field is maximally complete [11] [21] [25]. $\overline{\kappa}((\overline{V}))(\sqrt{-1}) = \overline{\kappa}(\sqrt{-1})((\overline{V}))$ is algebraically closed, so $\overline{\kappa}((\overline{V}))$ is real closed. The natural valuation on $\overline{\kappa}((\overline{V}))$, denoted also by v, is the unique finest valuation on $\overline{\kappa}((\overline{V}))$ compatible with the ordering. The following is a summary of our results.

Suppose a proper embedding (see Section 1 for the definition) $p : \overline{F} \to \overline{\kappa}((\overline{V}))$ is given and F' is an ordered extension of F generated by a single element y. We show there exists a canonically defined power series $\phi \in \overline{\kappa'}((\overline{V'}))$ such that p extends to a proper embedding $p : \overline{F'} \to \overline{\kappa'}((\overline{V'}))$ via $y \mapsto \phi$. Using results of Mourgues and Ressayre [16] we show that $p(\overline{F})$ truncation closed $\Rightarrow p(\overline{F'})$ truncation closed. We also show that $p(F) \subseteq \widetilde{\kappa}((V)) \Rightarrow \phi \in \widetilde{\kappa'}((V'))$ ($\widetilde{\kappa}$ denotes the smallest subextension $\kappa \subseteq \widetilde{\kappa} \subseteq \overline{\kappa}$ closed under adjoining n-th roots of positive elements, $n \geq 1$. Of course, if $\kappa = \mathbb{R}$, then $\kappa = \widetilde{\kappa} = \overline{\kappa}$). Roughly, this is what is asked for by MacLane and Schilling in [14, Final Remark]. It allows us to "read off" the extended value group V' directly from the power series ϕ.

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In Section 5 we apply this result to show that, modulo natural constraints, the extended value group V' can be prescribed arbitrarily.

In Sections 3 and 4 we point out a natural one-to-one correspondence between orderings on the polynomial ring $\overline{\kappa}[Y_1, ..., Y_\ell]$, κ a given archimedean ordered field, and certain distinguished ℓ-tuples $(\phi_1, ..., \phi_\ell)$, $\phi_i \in \mathbb{R}((\mathbb{H}_i))$, $i = 1, ..., \ell$ where $\mathbb{H}_0 \hookrightarrow ... \hookrightarrow \mathbb{H}_\ell$, $\mathbb{H}_i$ is the product of $2^i - 1$ copies of $(\mathbb{R}, +, \leq)$ ordered lexicographically, and "read off" properties of the ordering from the associated ℓ-tuple. Orderings on $\overline{\kappa}[Y_1, ..., Y_\ell]$ also correspond to ultrafilters of semialgebraic sets in $\overline{\kappa}^\ell$ [4] [5] so another way of describing this result is to say that we have associated a distinguished "ideal point" $(\phi_1, ..., \phi_\ell)$ to each such ultrafilter.

In Section 6, we apply the analysis in Sections 1 and 2 to investigate which cuts are realized in a real closed field. In particular we give a valuation theoretic characterization of $\aleph_\alpha$–saturated real closed fields, for a cardinal $\aleph_\alpha \geq \aleph_0$ (cf. Theorem 6.2 below).

The results of Sections 1 and 2 are needed throughout the remainder of the paper, and Section 4 depends on Section 3 as well. Otherwise, the sections are independent of each other. After the first two sections, the reader could continue either with Sections 3 and 4, or with Section 5, or go directly to Section 6, depending on the interest.

The authors want to thank M.E. Alonso and C. Andradas, for sharing their expertise with us. The original motivation comes from the paper [3] by M.E. Alonso, J.M. Gamboa and J.M. Ruiz. The focus in [3] (also see [2]) is on describing orderings on $\mathbb{R}(Y_1, ..., Y_\ell)$ in terms of analytic germs $\psi : (0, \epsilon) \to \mathbb{R}^\ell$.

1. The Embedding Theorem.

We use the terminology and notation introduced above. Any embedding $p : \overline{F} \to \overline{\kappa}((\overline{V}))$ is order preserving and consequently also valuation preserving. Since $\overline{\kappa}$ is archimedean, the induced embedding of the residue fields $\overline{\kappa} \hookrightarrow \overline{\kappa}$ is the identity mapping. We say that an embedding $p : \overline{F} \to \overline{\kappa}((\overline{V}))$ is *proper* if the induced embedding of value groups $\overline{V} \hookrightarrow \overline{V}$ is the identity mapping, i.e., if $v(p(z)) = v(z)$ for all $z \in \overline{F}$. The *support* of a power series $a = \sum_{\gamma \in \overline{V}} a_\gamma x^\gamma \in \overline{\kappa}((\overline{V}))$ is the set $\{\gamma \in \overline{V} \mid a_\gamma \neq 0\}$. This is well-ordered by definition. A power series $b = \sum_{\gamma \in \overline{V}} b_\gamma x^\gamma$ is a *truncation* of a power series $a = \sum_{\gamma \in \overline{V}} a_\gamma x^\gamma$ if there exists $\gamma_0 \in \overline{V} \cup \{\infty\}$ such that

$$b_\gamma = \begin{cases} a_\gamma & \text{if } \gamma < \gamma_0 \\ 0 & \text{if } \gamma \geq \gamma_0 \end{cases}.$$

We write $b = (a)_{<\gamma_0}$ in this case. A subfield K of a power series field $\overline{\kappa}((\overline{V}))$ is said to be *truncation closed* if every truncation of every element of K belongs to K [16].

1.1 Theorem. *Suppose $F' = F(y)$, an ordered extension of F, and $p : \overline{F} \to \overline{\kappa}((\overline{V}))$ is a proper embedding. Then there is a canonically defined power series $\phi \in \overline{\kappa'}((\overline{V'}))$ (depending on y) such that:*

(1) p extends to a proper embedding $p : \overline{F'} \to \overline{\kappa'}((\overline{V'}))$ via $y \mapsto \phi$.

(2) If $p(\overline{F})$ is truncation closed, then so is $p(\overline{F'})$.

(3) If $p(F) \subseteq \widetilde{\kappa}((V))$, then $\phi \in \widetilde{\kappa'}((V'))$. Equivalently, $p(F') \subseteq \widetilde{\kappa'}((V'))$; equivalently, $V' = V_1$ and $\widetilde{\kappa'} = \widetilde{\kappa_1}$ where V_1 is the group generated by V and the support of ϕ and κ_1 is the field generated by κ and the coefficients of ϕ.

The definition of ϕ and the proof of part (1) of Theorem 1.1 (see below) seem to be part of the folklore. Also, as we will show, (2) is a consequence of the results in [16]. In this sense, the main new result here is (3). It does not seem possible to improve on (3) in any substantial way. For example, it is not true in general that $p(F) \subseteq \kappa((V)) \Rightarrow \phi \in \kappa'((V'))$.

The equivalence of the various conclusions in (3) is pretty clear: Since $p(F') = p(F(y)) = p(F)(\phi)$ and $p(F) \subseteq \widetilde{\kappa}((V))$, it is clear that $p(F') \subseteq \widetilde{\kappa'}((V'))$ iff $\phi \in \widetilde{\kappa'}((V'))$. From the definition of V_1 and κ_1 it is clear that $p(F') \subseteq \widetilde{\kappa_1}((V_1))$ so, comparing value groups and residue fields, $V' \subseteq V_1$ and $\kappa' \subseteq \widetilde{\kappa_1}$ (so $\widetilde{\kappa'} \subseteq \widetilde{\kappa_1}$). If $p(F') \subseteq \widetilde{\kappa'}((V'))$ then we obviously have the reverse inclusions, so $V' = V_1$ and $\widetilde{\kappa'} = \widetilde{\kappa_1}$. Conversely, if $V' = V_1$ and $\widetilde{\kappa'} = \widetilde{\kappa_1}$ then $p(F') \subseteq \widetilde{\kappa_1}((V_1)) = \widetilde{\kappa'}((V'))$.

We have the following easy consequence of Theorem 1.1 (cf. also [20], Satz 27, p. 118, and Satz 34, p. 124):

1.2 Corollary. *There exists a proper embedding $p : \overline{F} \to \overline{\kappa}((\overline{V}))$ such that*

(1) *$p(\overline{F})$ is truncation closed.*

(2) *$p(F) \subseteq \widetilde{\kappa}((V))$.*

Proof. This is immediate from Theorem 1.1, using Zorn's Lemma. □

1.3 Definition. The definition of ϕ is complicated by the fact that there are four cases to consider. If y is algebraic over F, there is a unique F-embedding $i : F(y) \to \overline{F}$ preserving the ordering. Define $\phi = p(i(y))$ in this case. We refer to this as the *algebraic case.* Suppose now that y is transcendental over F. The ordering on $F(y)$ extends uniquely to $\overline{F(y)}$ and, consequently, to the intermediate field $\overline{F}(y)$. Let

$$S = \{v(y - z) \mid z \in \overline{F}\}.$$

We define an element w in $\overline{\kappa}((\overline{V}))$ as follows: Suppose $\gamma \in \overline{V}$. If $\gamma \geq \delta$ for all $\delta \in S$ define $w_\gamma = 0$. If $\gamma < \delta = v(y-z)$ for some $z \in \overline{F}$, define w_γ to be the coefficient of x^γ in $p(z)$.

(Observe: w_γ is independent of the choice of z. If we also have $\gamma < \delta' = v(y-z')$, $z' \in \overline{F}$, then $v(p(z)-p(z')) = v(z-z') \geq \min\{\delta, \delta'\} > \gamma$ so the coefficient of x^γ in $p(z)$ and $p(z')$ are the same.) Define $w = \sum_{\gamma \in \overline{V}} w_\gamma x^\gamma$.

In what follows, $p(z)_\gamma$ denotes the coefficient of x^γ in $p(z)$. For $z \in \overline{F}$, $p(z) \in \overline{\kappa}((\overline{V}))$ so $\{\gamma \in \overline{V} \mid p(z)_\gamma \neq 0\}$ is well-ordered. Consequently, $\{\gamma \in \overline{V} \mid w_\gamma \neq 0\}$ is also well-ordered so w is indeed an element of $\overline{\kappa}((\overline{V}))$. Note that $w_\gamma = p(z)_\gamma$ for all $\gamma < v(y-z)$. Hence, $v(w-p(z)) \geq v(y-z)$.

Either S has no largest element or S has a largest element. If S has no largest element, we define $\phi = w$. We refer to this as the *immediate transcendental case.*

Claim 1. In this case, $w \notin p(\overline{F})$. For, if $z \in \overline{F}$, there exists $z' \in \overline{F}$ such that $v(y-z) < v(y-z')$. Then $v(p(z')-p(z)) = v(z'-z) = v(y-z)$ and $v(w-p(z')) \geq v(y-z') > v(y-z)$. Consequently, $v(w-p(z)) = v(y-z) \neq \infty$ so $w \neq p(z)$.

This case does not occur if $p(\overline{F}) = \overline{\kappa}((\overline{V}))$. Suppose now that S has a largest element γ. Thus there exists $z \in \overline{F}$ such that $v(y-z) = \gamma$ and w is a truncation of $p(z)$. There are two subcases: Either $\gamma \in \overline{V}$ or $\gamma \notin \overline{V}$. Suppose $\gamma \notin \overline{V}$. In this case we define $\phi = w + x^\gamma$ if $y > z$ and $\phi = w - x^\gamma$ if $y < z$.

Claim 2. This definition does not depend on the choice of z. If we also have $z' \in \overline{F}$, $v(y-z') = \gamma$ then $v(z-z') \geq \gamma$. Since $\gamma \notin \overline{V}$, this forces $v(z-z') > \gamma$, so $y-z'$ and $y-z$ have the same sign.

We refer to this as the *value transcendental case.* Finally, suppose $\gamma \in \overline{V}$.

Claim 3. In this case, there is a unique element $r \in \mathbb{R} \backslash \overline{\kappa}$ such that if $z \in \overline{F}$, $v(y-z) = \gamma$, then $z < y$ iff $p(z) < w + rx^\gamma$ and $z > y$ iff $p(z) > w + rx^\gamma$. To prove this, fix $z_0 \in \overline{F}$ such that $v(y-z_0) = \gamma$ and $a \in \overline{F}$ such that $v(a) = \gamma$, $a > 0$. Let $r = p(z_0)_\gamma + sp(a)_\gamma$ where $s \in \overline{\kappa'}$ is the image of $\frac{y-z_0}{a}$ in the residue field. Since γ is the maximal element of S, $s \notin \overline{\kappa}$, so $r \notin \overline{\kappa}$. (If $s \in \overline{\kappa}$ then s is the image of some $b \in \overline{F}$, $v(b) = 0$. Then $v(\frac{y-z_0}{a} - b) > 0$ so $v(y-(z_0+ab)) > \gamma$, contradicting the definition of γ.) Suppose $z \in \overline{F}$, $v(y-z) = \gamma$. If $z < y$, then $\frac{y-z}{a} = \frac{y-z_0}{a} + \frac{z_0-z}{a} > 0$ so, going to the residue field, $s + \frac{p(z_0)_\gamma - p(z)_\gamma}{p(a)_\gamma} > 0$, i.e., $p(z)_\gamma < r$, so $p(z) < w + rx^\gamma$. Similarly, if $z > y$, then $p(z)_\gamma > r$ so $p(z) > w + rx^\gamma$. Finally, if we take $z = z_0 + ta$, $t \in \mathbb{Q}$, we see that as t runs through $\mathbb{Q}$, $p(z)_\gamma = p(z_0)_\gamma + tp(a)_\gamma$ runs through a dense subset of $\overline{\kappa}$. This implies r is unique.

In this case, define $\phi = w + rx^\gamma$. We refer to this case as the *residue transcendental case.* This case does not occur if $\overline{\kappa} = \mathbb{R}$.

2. Proof of Theorem 1.1.

We prove part (1) of Theorem 1.1. Every order embedding extends uniquely to the real closure so it suffices to prove that the extension of p to F' given by $y \mapsto \phi$ is an order embedding of F' into $\overline{\kappa'}((\overline{V'}))$. This is clear if y is algebraic. If y is transcendental, one uses the fact that orderings on $\overline{F}(y)$ are completely determined by the associated cuts in $\overline{F}$; see [7]. The terminology we are using is the following:

2.1 Terminology. A *lower cut* in an ordered set $(I, \leq)$ is a subset L of I such that $i \leq j$ and $j \in L \Rightarrow i \in L$. The associated *upper cut* in I is $U = I \backslash L$. A *cut* in I is a pair (L, U) where L is a lower cut and U is the associated upper cut.

The lower cut in $\overline{F}$ determined by the ordering on $\overline{F}(y)$ is $\{z \in \overline{F} \mid z < y\}$. Consequently, to prove (1) we are reduced to showing that, for $z \in \overline{F}$, $z < y$ iff $p(z) < \phi$ and $z > y$ iff $p(z) > \phi$. First suppose there exists $z' \in \overline{F}$ such that $v(y - z) < v(y - z')$. Then $z' - z = (y - z) - (y - z')$ has the same sign as $y - z$. Of course, $p(z') - p(z) = p(z' - z)$ has the same sign as $z' - z$. Also, $v(w - p(z')) \geq v(y - z')$ and $v(\phi - w) \geq v(y - z')$ so $v(\phi - p(z')) \geq v(y - z') > v(y - z) = v(z' - z) = v(p(z') - p(z))$ so $p(z') - p(z) = (\phi - p(z)) - (\phi - p(z'))$ has the same sign as $\phi - p(z)$. Combining these things we see that $\phi - p(z)$ has the same sign as $y - z$ as required. The other case is where S has a largest element γ and $v(y - z) = \gamma$. If $\gamma \notin \overline{V}$, then $p(z) = w +$ terms of value $> \gamma$ so, for all such z, either $y > z$ and $\phi = w + x^\gamma > p(z)$ or $y < z$ and $\phi = w - x^\gamma < p(z)$. Finally, if $\gamma \in \overline{V}$ then $\phi = w + rx^\gamma$ and the result is clear from the definition of r.

We also need to check that the extended p is proper. This is a consequence of the following case-by-case analysis:

2.2 Note. (1) In the algebraic case, $\overline{V'} = \overline{V}$, $\overline{\kappa'} = \overline{\kappa}$ and everything is clear. (2) In the immediate transcendental case, $\phi = w \in \overline{\kappa}((\overline{V}))$ so $p(\overline{F'}) \subseteq \overline{\kappa}((\overline{V}))$. The induced embedding $\overline{V'} \hookrightarrow \overline{V}$ is the identity on $\overline{V}$. Thus $\overline{V'} = \overline{V}$, the extended p is proper and $\overline{\kappa'} = \overline{\kappa}$. (3) In the residue transcendental case, $\phi = w + rx^\gamma$ so $p(\overline{F'}) \subseteq \overline{\kappa(r)}((\overline{V}))$ and $r \in \overline{\kappa'}$. Thus $\overline{V'} = \overline{V}$, the extended p is proper and $\overline{\kappa'} = \overline{\kappa(r)}$. (4) In the value transcendental case, $\phi = w \pm x^\gamma$ so $p(\overline{F'}) \subseteq \overline{\kappa}((\overline{V} \oplus \mathbb{Q}\gamma))$. The induced embedding $\overline{V'} \hookrightarrow \overline{V} \oplus \mathbb{Q}\gamma$ is the identity on $\overline{V}$ and sends γ to γ (since if $z \in \overline{F}$ is such that $v(y - z) = \gamma$, then $p(y - z) = \phi - p(z) = \pm x^\gamma +$ terms of greater value, so the image of γ is $v(p(y - z)) = \gamma$). Thus $\overline{V'} = \overline{V} \oplus \mathbb{Q}\gamma$, the extended p is proper in this case too and $\overline{\kappa'} = \overline{\kappa}$.

Thus the extension of p to $\overline{F'}$ is order preserving and proper in all cases so the proof of part (1) of Theorem 1.1 is complete. $\square$

In proving parts (2) and (3) of Theorem 1.1, there is no harm in identifying $\overline{F}$ with $p(\overline{F})$ and y with ϕ. It is also convenient, for later work, to have the following intrinsic description of ϕ.

2.3 Definition. Suppose the embedding $\overline{F} \subseteq \overline{\kappa}((\overline{V}))$ is proper. We say an element $\phi \in \mathbb{R}((H))$, H some ordered group extension of $\overline{V}$, is a *distinguished power series* for $\overline{F}$ if one of the following holds:

(1) $\phi \in \overline{F}$.
(2) $\phi \in \overline{\kappa}((\overline{V}))$, $\phi \notin \overline{F}$ and every proper truncation $(\phi)_{<\gamma}$ of ϕ is of the form $(\phi)_{<\gamma} = (c)_{<\gamma}$ for some $c \in \overline{F}$.
(3) $\phi = w \pm x^{\gamma}$, $\gamma \notin \overline{V}$ and $w = (c)_{<\gamma}$ for some $c \in \overline{F}$.
(4) $\phi = w + rx^{\gamma}$ where $r \in \mathbb{R}\backslash\overline{\kappa}$, $\gamma \in \overline{V}$ and $w = (c)_{<\gamma}$ for some $c \in \overline{F}$.

2.4 Theorem. *Suppose $\overline{F} \subseteq \overline{\kappa}((\overline{V}))$ is proper. If $F' = F(y)$ is an ordered extension of F then the canonically defined power series $\phi \in \overline{\kappa'}((\overline{V'}))$ corresponding to y is distinguished for $\overline{F}$. Conversely, if $\psi \in \mathbb{R}((H))$ is distinguished for $\overline{F}$, H some ordered group extension of $\overline{V}$, then there exists an ordered extension $F' = F(y)$ of F such that ψ is the canonically defined power series corresponding to y.*

Proof. The first assertion is clear from Definition 1.3. For the second, take $y = \psi$ and order $F' = F(y)$ via the embedding $F' \subseteq \mathbb{R}((H))$. We complete the proof by checking that the canonically defined power series ϕ corresponding to y is equal to y. Let $S = \{v(y - z) \mid z \in \overline{F}\}$.

(1) Suppose $y \in \overline{F}$, i.e., $\infty \in S$. Then clearly $\phi = y$.

(2) Suppose $y \in \overline{\kappa}((\overline{V}))$, $y \notin \overline{F}$, but every proper truncation $(y)_{<\gamma}$ of y is of the form $(y)_{<\gamma} = (z)_{<\gamma}$ for some $z \in \overline{F}$. Then S has no largest element. (If $v(y - z) = \gamma$, $z \in \overline{F}$, then picking $t \in \overline{F}$, $v(t) = \gamma$, $t = (y_\gamma - z_\gamma)x^{\gamma}+$ terms of higher value, we see that $z' = z + t \in \overline{F}$ satisfies $v(y - z') > \gamma$.) If $\gamma < v(y - z)$ for some $z \in \overline{F}$ then, if $\gamma \notin \overline{V}$, then $y_\gamma = z_\gamma = 0 = w_\gamma$, and, if $\gamma \in \overline{V}$, then $y_\gamma = z_\gamma = w_\gamma$. If $\gamma > S$ and $y_\gamma \neq 0$, then $(y)_{<\gamma}$ is a proper truncation of y, so $(y)_{<\gamma} = (z)_{<\gamma}$ for some $z \in \overline{F}$ so $v(y - z) \geq \gamma$, contradicting the definition of S. Thus $y_\gamma = w_\gamma = 0$ for $\gamma > S$. Thus $\phi = w = y$.

(3) Suppose that $y = w + \epsilon x^{\gamma}$, $\epsilon = \pm 1$, $\gamma \notin \overline{V}$, and $w = (z)_{<\gamma}$ for some $z \in \overline{F}$. Then $v(y - z) \geq \gamma$. Also, for any $z \in \overline{F}$, $z_\gamma = 0 \neq y_\gamma = \pm 1$, so $v(y - z) \leq \gamma$.

Thus γ is the largest element of S. Also, for any $z \in \overline{F}$ with $v(y-z) = \gamma$, $z < y$ iff $\epsilon = 1$ and $z > y$ iff $\epsilon = -1$. Thus $\phi = w + \epsilon x^\gamma = y$.

(4) Suppose that $y = w + rx^\gamma$, $r \in \mathbb{R}\backslash\kappa$, $\gamma \in \overline{V}$, and $w = (z)_{<\gamma}$ for some $z \in \overline{F}$. Then again, γ is the largest element of S and, if $z \in \overline{F}$, $v(y-z) = \gamma$, then $z < y$ iff $z_\gamma < r$ and $z > y$ iff $z_\gamma > r$. Thus $\phi = w + rx^\gamma = y$. $\square$

We return to the proof of Theorem 1.1. To simplify notation, identify $\overline{F}$ with $p(\overline{F})$ and y with ϕ. From the way ϕ is defined we see that every proper truncation of ϕ is the truncation of an element of $\overline{F}$. If $\overline{F}$ is truncation closed then every proper truncation of ϕ belongs to $\overline{F}$ so, by [16, Lemma 3.4], $K = \overline{F}(\phi)$ is truncation closed. In particular, $\kappa' \subseteq K$ so the field L obtained by adjoining to K all elements of $\overline{\kappa'}$ is algebraic over K. Also, any proper truncation of an element of $\overline{\kappa'}$ is $0 \in K$ so, by repeated application of [16, Lemma 3.4], L is truncation closed. Thus $x^\gamma \in L$ for all $\gamma \in V'$ so the field M obtained by adjoining to L all elements $x^{\gamma/n}$, $\gamma \in V'$, $n \geq 1$, is algebraic over L and, as before, is truncation closed. M has value group $\overline{V'}$ and residue field $\overline{\kappa'}$ so, by [16, Lemma 3.5], $\overline{M}$ is truncation closed. Since M is algebraic over K, $\overline{M} = \overline{F'}$. This completes the proof of part (2) of Theorem 1.1.

Suppose now that $F \subseteq \widetilde{\kappa}((V))$. We want to show that $\phi \in \widetilde{\kappa'}((V'))$ or, equivalently, that $F(\phi) \subseteq \widetilde{\kappa'}((V'))$. Let F^h denote the henselization of F and let $F(\phi)^h$ denote the henselization of $F(\phi)$ [21]. We view F^h and $F(\phi)^h$ as valued subfields of the big algebraically closed valued field $\mathbb{C}((\overline{V'}))$. Since $F \subseteq \widetilde{\kappa}((V))$ and $\widetilde{\kappa}((V))$ is henselian, $F^h \subseteq \widetilde{\kappa}((V))$. Similarly, since $F(\phi) \subseteq \overline{\kappa'}((\overline{V'}))$ and $\overline{\kappa'}((\overline{V'}))$ is henselian, $F(\phi)^h \subseteq \overline{\kappa'}((\overline{V'}))$. Also, $F \subseteq F(\phi)$ so $F^h \subseteq F(\phi)^h$.

Claim 1. If $c \in F(\phi)^h$ is algebraic over F then $c \in \widetilde{\kappa'}((V'))$. c is algebraic over F^h and F^h is henselian so, by [21, Th 2, p 236], the degree of $F^h(c)$ over F^h is equal to the product of the ramification index and the residue degree. Choose $d \in F^h(c)$ such that $F^h(d)$ is unramified over F^h and $F^h(c)$ is totally ramified over $F^h(d)$. We can assume the minimal polynomial f of d over F^h is monic with coefficients in the valuation ring of F^h. Since $F(\phi)^h$ is an immediate extension of $F(\phi)$, the image of d in the residue field, call this $\overline{d}$, lies in κ'. Since $F^h \subseteq \widetilde{\kappa}((V))$, the coefficients of f are also in $\widetilde{\kappa}((V))$. By the existence assertion of Hensel's Lemma there exists a root e of f in $\widetilde{\kappa'}((V))$ such that $\overline{e} = \overline{d}$. By the uniqueness assertion of Hensel's Lemma, $e = d$. Thus $d \in \widetilde{\kappa'}((V))$ so $F^h(d) \subseteq \widetilde{\kappa'}((V))$. Suppose γ is in the value group of $F^h(c)$ and $y \in F^h(c)$ is chosen so that $v(y) = \gamma$. Replacing y by $-y$, we can assume $y > 0$. There exists an integer $n \geq 1$ and $z \in F^h(d)$ such that $y^n - z$ has value greater than

$n\gamma$. Thus $z = y^n(1+t)$ where $t \in F^h(c)$ has value > 0. $F^h(c)$ is also henselian so $(1+t)^{1/n} \in F^h(c)$. Thus $z^{1/n} = y(1+t)^{1/n} \in F^h(c)$, and $v(z^{1/n}) = \gamma$. On the other hand, $z \in F^h(d) \subseteq \widetilde{\kappa'}((V))$ so $z = ax^{n\gamma}(1+p)$ for some $p \in \widetilde{\kappa'}((V))$ of positive value and $a \in \widetilde{\kappa'}$, $a > 0$. Thus $z^{1/n} = a^{1/n}x^{\gamma}(1+p)^{1/n} \in \widetilde{\kappa'}((V'))$. Applying this to a set of generators $\gamma_1, ..., \gamma_k$ of the value group of $F^h(c)$ modulo V, we get elements $z_i \in F^h(d)$ with $z_i^{1/n_i} \in \widetilde{\kappa'}((V'))$, $i = 1, ..., k$, and $F^h(c) = F^h(d, z_1^{1/n_1}, ..., z_k^{1/n_k})$. This implies $F^h(c) \subseteq \widetilde{\kappa'}((V'))$.

This completes the algebraic case. If $\phi \in \overline{F}$, then ϕ is algebraic over F so by Claim 1, $\phi \in \widetilde{\kappa'}((V'))$. Suppose now that $\phi \notin \overline{F}$. To show $\phi \in \widetilde{\kappa'}((V'))$ in this case, it suffices to show the following:

Claim 2. For each $\gamma \in S$, there exists $c \in F(\phi)^h$ algebraic over F with $v(\phi - c) \geq \gamma$.

For, by Claim 2 and the definition of w, each term $w_\delta x^\delta$ appearing in w is a term of some $c \in F(\phi)^h$ which is algebraic over F so, by Claim 1, $\delta \in V'$ and $w_\delta \in \widetilde{\kappa'}$. Thus $w \in \widetilde{\kappa'}((V'))$. If S has no largest element then $\phi = w \in \widetilde{\kappa'}((V'))$. If S has a largest element γ then, by Claim 2, there exists $c \in F(\phi)^h$ algebraic over F such that $v(\phi - c) \geq \gamma$. Since $c \in \overline{F}$, the definition of S implies $v(\phi - c) = \gamma$. (Recall: y and ϕ are identified now.) Since $\phi - c \in F(\phi)^h$, an immediate extension of $F(\phi)$, this implies $\gamma \in V'$. Thus, if $\gamma \notin \overline{V}$ then $\phi = w \pm x^\gamma \in \widetilde{\kappa'}((V'))$. Suppose now that $\gamma \in \overline{V}$ so $\phi = w + rx^\gamma$ for some $r \in \mathbb{R}\backslash\overline{\kappa}$. Pick an integer $n \geq 1$ such that $n\gamma \in V$ and $b \in F$ with $v(b) = n\gamma$. Then $v(\frac{(\phi-c)^n}{b}) = 0$ and $\frac{(\phi-c)^n}{b} \in F(\phi)^h$, an immediate extension of $F(\phi)$, so the image of $\frac{(\phi-c)^n}{b}$ in the residue field belongs to κ' and has the form $u = \frac{(r-s)^n}{t}$ where s is the coefficient of x^γ in c and t is the coefficient of $x^{n\gamma}$ in b. Thus $t \in \widetilde{\kappa}$ and $s \in \widetilde{\kappa'}$ (by Claim 1) so $r \in \widetilde{\kappa'}$. Thus $\phi = w + rx^\gamma \in \widetilde{\kappa'}((V'))$ is this case too.

To prove Claim 2, fix $\gamma \in S$ and $d \in \overline{F}$ such that $v(\phi - d) = \gamma$. Let $H = F(\phi)^h \cap \overline{F}$. H is an algebraic extension of F^h so is henselian. Let e and f denote the ramification index and residue degree of $H(d)$ over H. Pick $1 = x_1, ..., x_e \in H(d)$ so that $v(x_1), ..., v(x_e)$ generate the distinct cosets of the value group extension and $1 = y_1, ..., y_f \in H(d)$ of value zero such that the residues $\overline{y_1}, ..., \overline{y_f}$ are a basis for the residue field extension. Thus ef is the degree of $H(d)$ over H and the x_iy_j, $i = 1, ..., e$, $j = 1, ..., f$ form a basis of $H(d)$ over H. Also, by our construction, $v(\sum_{i,j} c_{ij}x_iy_j) = \min_{i,j}\{v(c_{ij}x_iy_j)\}$ for any $c_{ij} \in H$. In particular, if $d = \sum_{i,j} c_{ij}x_iy_j$ then $c = c_{11}$ is a 'best approximation' of d in H in the sense that $v(d - c) \geq v(d - c')$ for any $c' \in H$. It remains to prove that $v(\phi - c) \geq \gamma$. Suppose this is not the case. Say $v(\phi - c) = \delta < \gamma$. Then

$\phi - c = qx^\delta + ...$, $\delta \in \overline{V}$, $q \neq 0$, $q \in \overline{\kappa}$. Then $n\delta \in V$ for some integer $n \geq 1$ so we have some element $a = tx^{n\delta} + ...$ in F, $t \in \widetilde{\kappa}$, $t \neq 0$. Thus $\phi - c$ and a belong to $F(\phi)^h$ so $(\phi - c)^n/a \in F(\phi)^h$, $v((\phi - c)^n/a) = 0$. The image of $(\phi - c)^n/a$ in the residue field is q^n/t. q and t belong to $\overline{\kappa}$ so q^n/t is in $\overline{\kappa}$ so is algebraic over the residue field κ of F. Thus, by Hensel's Lemma applied to $F(\phi)^h$, we have an element b in the valuation ring of $F(\phi)^h$, algebraic over F and such that the image of b in the residue field is q^n/t. Thus $(\phi - c)^n - ab$ has value $> n\delta$ so $ab = (\phi - c)^n(1+p)$ where $p \in F(\phi)^h$ has value > 0. Then $(1+p)^{1/n} \in F(\phi)^h$ and $((\phi - c)(1+p)^{1/n})^n = ab$ is algebraic over F, so $(\phi - c)(1+p)^{1/n}$ is algebraic over F. Take $c' = c + (\phi - c)(1+p)^{1/n}$. Thus $v(\phi - c') > v(\phi - c)$ and $v(\phi - d) > v(\phi - c)$ so $v(d - c') > v(\phi - c) = v(d - c)$. Since $c' \in H$, this contradicts the definition of c. $\square$

3. Orderings on the polynomial ring $\overline{\kappa}[Y_1, ..., Y_\ell]$.

Let A be a commutative ring with 1. By an *ordering* on A we mean a subset P of A such that $P + P \subseteq P$, $PP \subseteq P$, $P \cup -P = A$, and $\mathfrak{p} := P \cap -P$ (called the support of P) is a prime ideal of A. Equivalently, an ordering on A is a pair $(\mathfrak{p}, \leq)$ where $\mathfrak{p}$ is a prime ideal of A and $\leq$ is an ordering on $\mathrm{qf}(A/\mathfrak{p})$, the quotient field of the domain $A/\mathfrak{p}$ [4] [5] [10] [15].

Consider the set-up of Theorem 1.1. In the algebraic, immediate transcendental and residue transcendental cases, ϕ is an element of $\mathbb{R}((\overline{V}))$. In the value transcendental case, ϕ has the form $\phi = w + \epsilon x^\gamma$ where $\epsilon = \pm 1$, $\gamma \notin \overline{V}$ belongs to some ordered group extension of $\overline{V}$ and $w \in \overline{\kappa}((\overline{V}))$ is obtained from an element of $\overline{F}$ by truncating the terms with value $\geq \gamma$. Unfortunately, the value group $\overline{V'} = \overline{V} \oplus \mathbb{Q}\gamma$ is only determined up to isomorphism over $\overline{V}$. If $\gamma_1 \notin \overline{V}$ is an element in some ordered group extension of $\overline{V}$ defining the same cut in $\overline{V}$ as the element γ, then $\gamma \mapsto \gamma_1$ defines an order isomorphism $\overline{V} \oplus \mathbb{Q}\gamma \cong \overline{V} \oplus \mathbb{Q}\gamma_1$ over $\overline{V}$ [23]. Thus, if we use this other group as the value group, then the power series we obtain changes from $\phi = w + \epsilon x^\gamma$ to $\phi_1 = w + \epsilon x^{\gamma_1}$. This is not very satisfactory. To get a canonical power series ϕ associated to each value transcendental ordering, we need a canonical way of choosing the elements γ we use to fill the various cuts in $\overline{V}$.

For $(I, \leq)$ a totally ordered set, let $H = \mathrm{H}(I)$ denote the Hahn group of rank I. By definition, H is the ordered Abelian group consisting of all functions $\alpha : I \to \mathbb{R}$ having well-ordered support with pointwise addition, ordered lexicographically. Denote by $v_s : H \to I \cup \{\infty\}$ the natural set valuation, i.e., $v_s(0) = \infty$ and, for $\alpha \in H$, $\alpha \neq 0$, $v_s(\alpha)$ is the least i such that $\alpha(i) \neq 0$ [6].

Let $\overline{I}$ denote the set of all cuts of I. The set $I' = I \,\dot\cup\, \overline{I}$ has a natural ordering so we can form the Hahn group $H' = \mathrm{H}(I')$. The natural embedding $I \hookrightarrow I'$ is order preserving and induces an ordered group embedding $H \hookrightarrow H'$ preserving the set valuation v_s. Here we are mainly interested in the case where I is finite. Note: if $|I| = n$ then $|\overline{I}| = n+1$ so $|I'| = 2n+1$. Thus, if we iterate the process, starting with $I = \emptyset$, we obtain a sequence of Hahn groups

$$\mathbb{H}_0 \hookrightarrow \mathbb{H}_1 \hookrightarrow ...$$

defined recursively by $\mathbb{H}_0 = \mathrm{H}(\emptyset) = \{0\}$, $\mathbb{H}_{i+1} = (\mathbb{H}_i)'$. We can identify $\mathbb{H}_i$ with the product of $2^i - 1$ copies of $(\mathbb{R}, +, \leq)$, ordered lexicographically. If we do this, the embedding $\mathbb{H}_i \hookrightarrow \mathbb{H}_{i+1}$ is given by

$$(r_1, ..., r_{2^i-1}) \mapsto (0, r_1, 0, ..., 0, r_{2^i-1}, 0).$$

3.1 Lemma. *Let $\overline{V}$ be a (divisible) subgroup of $\mathbb{H}_i$. Then each cut (S_1, S_2) in $\overline{V}$ is filled by a canonically defined element γ in $\mathbb{H}_{i+1}$.*

Proof. The definition of γ is analogous to the definition of ϕ given in Section 1. Let

$$T = \{v_s(\alpha_2 - \alpha_1) \mid \alpha_1 \in S_1, \alpha_2 \in S_2\}.$$

There is a unique element δ in $\mathbb{H}_i$ such that $\delta(j) = 0$ if $j > v_s(\alpha_2 - \alpha_1)$ for all $\alpha_1 \in S_1$ and all $\alpha_2 \in S_2$, $\delta(j) = \alpha_1(j)$ if $j < v_s(\alpha_2 - \alpha_1)$ for some $\alpha_1 \in S_1$ and some $\alpha_2 \in S_2$ and, if j is the largest element of T, then $\alpha_1(j) \leq \delta(j) \leq \alpha_2(j)$ for all $\alpha_1 \in S_1$ and all $\alpha_2 \in S_2$ satisfying $v_s(\alpha_1 - \alpha_2) = j$. Note: if $T = \emptyset$, i.e., if one of S_1, S_2 is empty, then $\delta = 0$. If there does not exist an element $\alpha \in \overline{V}$ such that

$$\delta(i) = \begin{cases} \alpha(i) & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases},$$

we take $\gamma = \delta$. If such an element $\alpha \in \overline{V}$ exists, then either all such α belong to S_1 or all such α belong to S_2. In this case, we take $\gamma = \delta \pm \chi_{\{n\}}$, with the '+' sign if all these α belong to S_1, and the '$-$' sign if they all belong to S_2, where $\chi_{\{n\}}$ denotes the characteristic function of $\{n\}$ and $n \in v_s(\mathbb{H}_{i+1})$ is chosen to fill the gap between the last element of T and the first element of $v_s(\overline{V})$ greater than the last element of T. (Note: This makes sense even if ∞ is the only element of $v_s(\overline{V})$ greater than the last element of T. It also makes sense if $T = \emptyset$.) There may be several such n. To get a canonical γ, look at the smallest k, $1 \leq k \leq i+1$ such that there is such an element n in $v_s(\mathbb{H}_k)$. Then, in $v_s(\mathbb{H}_k)$ there is exactly one such element n, and this is the one we pick. $\square$

Fix an archimedean ordered field κ. Using Lemma 3.1, we will see now that orderings on the polynomial ring $\overline{\kappa}[Y_1, ..., Y_\ell]$ are in natural one-to-one correspondence with certain distinguished ℓ-tuples of the form $(\phi_1, ..., \phi_\ell)$ where $\phi_i \in \mathbb{R}((\mathbb{H}_i))$, $i = 1, ..., \ell$.

These are defined recursively. Suppose that $(\phi_1, ..., \phi_j)$ has been defined, $0 \leq j < \ell$, $F_j \subseteq \widetilde{\kappa_j}((V_j))$ and $\overline{F_j}$ is truncation closed in $\overline{\kappa_j}((\overline{V_j}))$, where $F_j = \kappa(\phi_1, ..., \phi_j)$, $V_j \subseteq \mathbb{H}_j$ denotes the value group of F_j and κ_j denotes the residue field of F_j. Then ϕ_{j+1} is any element of the set $\cup_{i=1}^4 S_i \subseteq \mathbb{R}((\mathbb{H}_{j+1}))$ where:

- $S_1 = \overline{F_j}$.
- S_2 consists of all $\phi \in \overline{\kappa_j}((\overline{V_j}))$ such that ϕ is not element of $\overline{F_j}$ but every proper truncation of ϕ is an element of $\overline{F_j}$.
- S_3 consists of all elements $\phi = w \pm x^\gamma$, γ the distinguished element of $\mathbb{H}_{j+1}$ corresponding to some cut in $\overline{V_j}$ (see Lemma 3.1), w an element of $\overline{F_j}$ having no terms of value $\geq \gamma$.
- S_4 consists of all elements $\phi = w + rx^\gamma$, $\gamma \in \overline{V_j}$, $r \in \mathbb{R}\backslash\overline{\kappa_j}$, w an element of $\overline{F_j}$ having no terms of value $\geq \gamma$.

In this way, $(\phi_1, ..., \phi_{j+1})$ is defined. Note: By Theorem 2.4 and parts (2) and (3) of Theorem 1.1, $\phi_{j+1} \in \widetilde{\kappa_{j+1}}((V_{j+1}))$ and $\overline{F_{j+1}}$ is truncation closed in $\overline{\kappa_{j+1}}((\overline{V_{j+1}}))$.

3.2 Remark. The sets S_1, S_2, S_3, S_4 correspond to the algebraic, immediate transcendental, value transcendental and residue transcendental cases respectively. If $\overline{\kappa} = \mathbb{R}$ then $\kappa_j = \widetilde{\kappa_j} = \overline{\kappa_j} = \mathbb{R}$ and the set S_4 corresponding to the residue transcendental case is the empty set.

The ordering on $\overline{\kappa}[Y_1, ..., Y_\ell]$ corresponding to the ℓ-tuple $(\phi_1, ..., \phi_\ell)$ is defined to be the one induced by the $\overline{\kappa}$-algebra homomorphism $\overline{\kappa}[Y_1, ..., Y_\ell] \to \mathbb{R}((\mathbb{H}_\ell))$, $Y_i \mapsto \phi_i$, $i = 1, ..., \ell$.

3.3 Theorem. *The correspondence defined above, between distinguished ℓ-tuples and orderings on $\overline{\kappa}[Y_1, ..., Y_\ell]$, is one-to-one and onto.*

Proof. Let $P_j = P \cap \overline{\kappa}[Y_1, ..., Y_j]$, $0 \leq j \leq \ell$ where P is an ordering of $\overline{\kappa}[Y_1, ..., Y_\ell]$. Suppose, by induction, that there is a unique distinguished j-tuple $(\phi_1, ..., \phi_j)$ defining P_j, $0 \leq j < \ell$. Let $F_j = \kappa(\phi_1, ..., \phi_j) \subseteq \widetilde{\kappa_j}((V_j))$. Extensions of P_j to $\overline{\kappa}[Y_1, ..., Y_{j+1}]$ are in one-to-one correspondence with orderings on $\overline{F_j}[Y_{j+1}]$. Orderings of $\overline{F_j}[Y_{j+1}]$ having support $\neq \{0\}$ correspond to elements of S_1. By Theorem 1.1 (1), Theorem 2.4 and Lemma 3.1, support $\{0\}$ orderings of $\overline{F_j}[Y_{j+1}]$

correspond to elements of $S_2 \cup S_3 \cup S_4$. Here, S_1, S_2, S_3, S_4 are defined as above. Thus we have a unique distinguished $j+1$-tuple $(\phi_1, ..., \phi_{j+1})$ defining P_{j+1}. $\square$

3.4 Notes. (1) The support of the ordering is the kernel of the corresponding mapping from $\overline{\kappa}[Y_1, ..., Y_\ell]$ to $\mathbb{R}((\mathbb{H}_\ell))$, call this $\mathfrak{p}_\ell$. $\operatorname{trdeg}(F_\ell : \kappa)$ is the depth of $\mathfrak{p}_\ell$ and $\ell-$ $\operatorname{trdeg}(F_\ell : \kappa)$ is the height of $\mathfrak{p}_\ell$. $\ell-$ $\operatorname{trdeg}(F_\ell : \kappa)$ also counts the number of algebraic ϕ_i occuring in $(\phi_1, ..., \phi_\ell)$. The following well-known inequality holds:

$$\operatorname{trdeg}(F_\ell : \kappa) \geq \ \dim_{\mathbb{Q}}(\overline{V_\ell}) + \ \operatorname{trdeg}(\kappa_\ell : \kappa).$$

$\dim_{\mathbb{Q}}(\overline{V_\ell})$ counts the number of value transcendental ϕ_i. $\operatorname{trdeg}(\kappa_\ell : \kappa)$ counts the number of residue transcendental ϕ_i. The difference

$$\operatorname{trdeg}(F_\ell : \kappa) - (\dim_{\mathbb{Q}}(\overline{V_\ell}) + \ \operatorname{trdeg}(\kappa_i : \kappa))$$

counts the number of immediate transcendental ϕ_i. These assertions are clear from Note 2.2.

(2) Orderings on $\overline{\kappa}(Y_1, ..., Y_\ell)$, i.e., support zero orderings on $\overline{\kappa}[Y_1, ..., Y_\ell]$, correspond to distinguished ℓ-tuples $(\phi_1, ..., \phi_\ell)$ where all of the ϕ_i are transcendental.

(3) The value group V_ℓ is generated by the union of the supports of the ϕ_i, $i = 1, ..., \ell$. $\widetilde{\kappa_\ell}$ is the smallest field containing κ and the coefficients of the ϕ_i and closed under taking n-th roots of positive elements, $n \geq 1$.

(4) Every ℓ-tuple $(\psi_1, ..., \psi_\ell)$, $\psi_i \in \mathbb{R}((\mathbb{H}_\ell))$, $i = 1, ..., \ell$, gives rise to a $\overline{\kappa}$-homomorphism $\overline{\kappa}[Y_1, ..., Y_\ell] \to \mathbb{R}((\mathbb{H}_\ell))$ via $Y_i \mapsto \psi_i$, $i = 1, ..., \ell$, and hence to an ordering on $\overline{\kappa}[Y_1, ..., Y_\ell]$. Let us view two such ℓ-tuples as being equivalent if they define the same ordering on $\overline{\kappa}[Y_1, ..., Y_\ell]$. By Theorem 3.3, for any such ℓ-tuple $(\psi_1, ..., \psi_\ell)$, there is a unique distinguished ℓ-tuple in the same class. Unfortunately, it seems very difficult to describe this distinguished ℓ-tuple in any simple way in terms of $\psi_1, ..., \psi_\ell$. In this sense, the problem poised by MacLane and Schilling in [11, Final Remark] remains unsolved.

(5) For example, the specializations of the ordering [4, p. 35] [5, p. 116] corresponding to a given distinguished ℓ-tuple $(\phi_1, ..., \phi_\ell)$ are naturally represented by (not necessarily distinguished) ℓ-tuples $(\phi'_1, ..., \phi'_\ell)$ where each ϕ'_i is a suitable truncation of ϕ_i, $i = 1, ..., \ell$, but it is difficult to describe explicitly the distinguished ℓ-tuples corresponding to these specializations.

3.5 Example. Orderings on $\overline{\kappa}[Y]$. Here, $\mathbb{H}_0 = \{0\}$, $\mathbb{H}_1 = \mathbb{R}$. In the algebraic case, $\phi = a$, $a \in \overline{\kappa}$. The immediate transcendental case does not occur. In the value transcendental case, either $\phi = \pm x^{-1}$ ($w = 0$, $\gamma = -1$) or $\phi = a \pm x$, $a \in \overline{\kappa}$ ($w = a$, $\gamma = 1$). In the residue transcendental case, $\phi = a$, $a \in \mathbb{R}\backslash\overline{\kappa}$ ($w = 0$, $\gamma = 0$, $r = a$).

4. Real places on $\overline{\kappa}[Y_1, ..., Y_\ell]$.

Define a 2-*character* of V_ℓ to be a group homomorphism $\sigma : V_\ell \to \{-1, 1\}$. Every 2-character σ of V_ℓ induces an $\mathbb{R}$-automorphism of the field $\mathbb{R}((V_\ell))$ given by $f = \sum a_\gamma x^\gamma \mapsto f_\sigma = \sum a_\gamma \sigma(\gamma) x^\gamma$.

4.1 Remark. Every 2-character on V_ℓ factors through the group $V_\ell/2V_\ell$. The group $V_\ell/2V_\ell$ is finite. If $\alpha_1 + 2V_\ell, ..., \alpha_n + 2V_\ell$ are $\mathbb{Z}/(2)$-independent, then $\alpha_1, ..., \alpha_n$ are $\mathbb{Q}$-independent. Thus $|V_\ell/2V_\ell| \leq 2^{\dim_{\mathbb{Q}}(\overline{V_\ell})}$. Consequently, the number of 2-characters of V_ℓ is equal to the order of the group $V_\ell/2V_\ell$.

Let A be a commutative ring with 1. A *real place* on A is defined to be a pair $(\mathfrak{p}, \lambda)$ where $\mathfrak{p}$ is a prime ideal of A and $\lambda : \mathrm{qf}(A/\mathfrak{p}) \to \mathbb{R} \cup \{\infty\}$ is a place. The real place associated to an ordering $(\mathfrak{p}, \leq)$ of A is $(\mathfrak{p}, \lambda)$ where λ is the real place on $\mathrm{qf}(A/\mathfrak{p})$ associated to the ordering $\leq$; see [10] [15] [19].

4.2 Corollary. *If $(\phi_1, ..., \phi_\ell)$ is a distinguished ℓ-tuple then, for each 2-character σ of V_ℓ, $(\phi_{1\sigma}, ..., \phi_{\ell\sigma})$ is a distinguished ℓ-tuple. The map $\sigma \mapsto (\phi_{1\sigma}, ..., \phi_{\ell\sigma})$ is injective. Orderings corresponding to the distinguished ℓ-tuples of the form $(\phi_{1\sigma}, ..., \phi_{\ell\sigma})$, σ a 2-character of V_ℓ, all belong to the same real place of $\overline{\kappa}[Y_1, ..., Y_\ell]$ and are the full equivalence class of orderings belonging to this real place.*

Proof. We prove, by induction on j, that $(\phi_{1\sigma}, ..., \phi_{j\sigma})$ is a distinguished j-tuple, $j = 0, ..., \ell$. Suppose $0 \leq j < \ell$, $F_j = \overline{\kappa}(\phi_1, ..., \phi_j)$, $F_{j\sigma} = \overline{\kappa}(\phi_{1\sigma}, ..., \phi_{j\sigma})$. Let V_j (resp., κ_j) denote the value group (resp., residue field) of F_j. The automorphism $f \mapsto f_\sigma$ is valuation preserving and maps F_j onto $F_{j\sigma}$ so V_j (resp., κ_j) is the value group (resp., residue field) of $F_{j\sigma}$. There are four cases. (1) ϕ_{j+1} is algebraic over F_j. Then $\phi_{(j+1)\sigma}$ is algebraic over $F_{j\sigma}$. (2) $\phi_{j+1} \in \overline{\kappa_j}((\overline{V_j}))$ is not algebraic over F_j but any proper truncation of ϕ_{j+1} is algebraic over F_j. Then $\phi_{(j+1)\sigma} \in \overline{\kappa_j}((\overline{V_j}))$ is not algebraic over $F_{j\sigma}$, but any proper truncation of $\phi_{(j+1)\sigma}$ is algebraic over $F_{j\sigma}$. (3) $\phi_{j+1} = w \pm x^\gamma$, w algebraic over F_j with no terms of value $\geq \gamma$, and γ is the distinguished element in $\mathbb{H}_{j+1}$ corresponding to some cut in $\overline{V_j}$. Then $\phi_{(j+1)\sigma} = w_\sigma \pm \sigma(\gamma) x^\gamma$, w_σ is algebraic over $F_{j\sigma}$ with no terms of value $\geq \gamma$, and γ is the distinguished element in $\mathbb{H}_{j+1}$ corresponding to some cut in $\overline{V_j}$. (4) $\phi_{j+1} = w + rx^\gamma$, w algebraic over F_j having no terms of value $\geq \gamma$, $\gamma \in \overline{V_j}$, $r \in \mathbb{R}\backslash\overline{\kappa_j}$. Then $\phi_{(j+1)\sigma} = w_\sigma + r\sigma(\gamma)x^\gamma$, w_σ is algebraic over $F_{j\sigma}$ having no terms of value $\geq \gamma$, $\gamma \in \overline{V_j}$, and $r\sigma(\gamma) = \pm r \in \mathbb{R}\backslash\overline{\kappa_j}$. Thus, in any case, $(\phi_{1\sigma}, ..., \phi_{(j+1)\sigma})$ is a distinguished $j+1$-tuple. The result follows by induction on j.

The real place associated to $(\phi_1, ..., \phi_\ell)$ is clearly the pair $(\mathfrak{p}_\ell, \lambda)$, where $\mathfrak{p}_\ell$ is the kernel of the $\overline{\kappa}$-homomorphism from $\overline{\kappa}[Y_1, ..., Y_\ell]$ to F_ℓ, $Y_i \mapsto \phi_i$, $i = 1, ..., \ell$, and λ is the composite place

$$\mathrm{qf}(\frac{\overline{\kappa}[Y_1, ..., Y_\ell]}{\mathfrak{p}_\ell}) \cong F_\ell \subseteq \mathbb{R}((V_\ell)) \stackrel{\mu}{\longrightarrow} \mathbb{R} \cup \{\infty\}$$

where the isomorphism is the $\overline{\kappa}$-isomorphism induced by $Y_i \mapsto \phi_i$, $i = 1, ..., \ell$ and μ is the canonical place on $\mathbb{R}((V_\ell))$. Since $\mu(f_\sigma) = \mu(f)$ for any $f \in \mathbb{R}((V_\ell))$, this is the same as the place associated to $(\phi_{1\sigma}, ..., \phi_{\ell\sigma})$.

V_ℓ is generated by the elements γ such that x^γ appears in some ϕ_i so, if σ, τ are distinct 2-characters of V_ℓ then $\phi_{j\sigma} \neq \phi_{j\tau}$ for some j so $(\phi_{1\sigma}, ..., \phi_{\ell\sigma}) \neq (\phi_{1\tau}, ..., \phi_{\ell\tau})$. The last assertion follows by counting, using the Baer-Krull Theorem, see [15, Theorem 1.3.2]. □

4.3 Examples. (1) Real places on $\overline{\kappa}(Y)$. In the value transcendental case, $V_1 = \mathbb{Z}$ so there are exactly two 2-characters of V_1. (Either $\sigma(1) = 1$ or $\sigma(1) = -1$.) $\phi = a + x$ and $\phi = a - x$ have the same associated real place. Also $\phi = x^{-1}$ and $\phi = -x^{-1}$ have the same associated real place. In the residue transcendental case, $V_1 = \{0\}$, a divisible group, so the only 2-character of V_1 is the trivial one.

(2) For the ordering on $\overline{\kappa}(Y_1, Y_2)$ corresponding to

$$\phi_1 = x,\ \phi_2 = \sum_{i=1}^{\infty} x^{1-1/i},$$

$V_1 = \mathbb{Z}$, $V_2 = \mathbb{Q}$. Since V_2 is divisible, the only 2-character of V_2 is the trivial one.

4.4 Remark. The results in Sections 3 and 4 carry over, in a suitably generalized form, to orderings on the polynomial ring $\overline{F}[Y_1, ..., Y_\ell]$, F an arbitrary ordered field; see [26].

5. Building extensions with prescribed value group.

We assume $F' = F(y)$, an ordered extension of F. We choose an embedding $p : \overline{F} \to \overline{\kappa}((\overline{V}))$ as in Corollary 1.2, and identify F with its image in $\widetilde{\kappa}((V))$. Our next result seems to be well-known. In any case, parts (1), (2) and (3) are well-known.

5.1 Theorem. *V' and κ' are restricted as follows:*

(1) Algebraic case: V'/V is finite and κ' is a finite extension of κ.

(2) Value transcendental case: $V' = \mathbb{Z}\delta \oplus W$ where $\mathbb{Z}\delta$ is infinite cyclic, $W \supseteq V$, W/V is finite, κ' is a finite extension of κ.

(3) Residue transcendental case: V'/V is finite, κ' is a rational function field in one variable over a finite extension of κ.

(4) Immediate transcendental case: V'/V is countable torsion, $\overline{\kappa}((V)) \not\subseteq \overline{F}$ if V'/V is finite, κ' is an algebraic extension of κ.

Proof. (1) is standard ([24], Corollary 2, p. 26, and Corollary, p. 52).

(2) Pick $b \in F'$ so that $v(b) \notin \overline{V}$. Then F' is a finite extension of $F(b)$, the value group of $F(b)$ is $V \oplus \mathbb{Z}\gamma$, $\gamma = v(b)$, and the residue field of $F(b)$ is κ. (2) is now clear, using (1).

(3) Pick $b \in F'$, $v(b) = 0$, so that the image of b in the residue field is transcendental over κ. Then F' is a finite extension of $F(b)$ and the value group of $F(b)$ is V; hence as in (1), V'/V is finite. By the Ruled Residue Theorem [17], κ' is a rational function field in one variable over a finite extension of κ.

(4) $\kappa' \subseteq \overline{\kappa}$ so κ' is algebraic over κ. For each $\gamma \in \overline{V}$ such that x^γ appears in ϕ, there exists $z \in \overline{F}$ with $v(\phi - z) > \gamma$. Thus if $W_\gamma =$ the group generated over V by the set of $\delta \in \overline{V}$ such that $\delta \leq \gamma$ and x^δ appears in ϕ, then W_γ/V is finite. This implies that V'/V is the union of an increasing chain of finite groups, so is countable. If V'/V is finite and $\overline{\kappa}((V)) \subseteq \overline{F}$ then $\overline{\kappa}((V')) \subseteq \overline{F}$, contradicting $\phi \notin \overline{F}$. □

5.2 Theorem. *Modulo the restrictions imposed by Theorem 5.1, the value group V' can be prescribed arbitrarily.*

Proof. In cases (1) (2) (3) below, W denotes an ordered group extension of V with W/V finite (so $W \subseteq \overline{V}$).

(1) Algebraic case. Suppose $W/V \cong \oplus_{i=1}^{s} \mathbb{Z}/(e_i)$. Let $\beta_i = e_i\alpha_i$, $i = 1, ..., s$ where $\alpha_1, ..., \alpha_s$ are elements of W corresponding to the natural generators of $\oplus_{i=1}^{s} \mathbb{Z}/(e_i)$, and let $F' = F(\sqrt[e_1]{p_1}, ..., \sqrt[e_s]{p_s})$ where $p_i \in F$ is such that $v(p_i) = \beta_i$, $p_i > 0$. Use the primitive element theorem to pick $q \in \overline{F}$ so that $F' = F(q)$. By our construction, $V' = W$.

(2) Value transcendental case. Suppose $U = \mathbb{Z}\delta \oplus W$, an ordered Abelian extension of W with $\mathbb{Z}\delta$ infinite cyclic. Pick $q \in \overline{F}$ so that the value group of $F(q)$ is W (as in Case 1). By Theorem 1.1, the exponents γ such that x^γ appears in q generate W over V. Since W/V is finite there is a element $\mu \in W$ such that the exponents γ such that x^γ appears in q and $\gamma \leq \mu$ generate W over V. Replacing

δ by $-\delta$ if necessary, we can assume $\delta > 0$. Replacing δ by $\delta + \mu$ if necessary, we can assume $\delta > \mu$. Let $\phi = w + x^\delta$ where w is obtained from q by truncating the terms in x^γ with $\gamma > \delta$. Then $F' = F(\phi)$ is value transcendental over F and, by Theorem 1.1 and Theorem 2.4, the value group of F' is $V' = \mathbb{Z}\delta \oplus W = U$.

(3) Residue transcendental case. Pick $q \in \overline{F}$ so that the value group of $F(q)$ is W (as in Case 1). Choose μ as in Case 2 and let $\phi = w + rx^\mu$ where w is obtained from q by truncating the terms in x^γ with $\gamma \geq \mu$ and $r \in \mathbb{R} \backslash \overline{\kappa}$ (if such r exists). Then $F' = F(\phi)$ is residue transcendental over F and $V' = W$ by Theorem 1.1 and Theorem 2.4.

(4) Immediate transcendental case. Suppose first that $V \subseteq W \subseteq \overline{V}$ and W/V is finite. Pick $q \in \overline{F}$ so the value group of $F(q)$ is W. Pick $q' \in \overline{\kappa}((V)) \backslash \overline{F}$ (if such an element exists). Multiplying q' by an element of F of sufficiently large value, we can assume the exponents γ such that x^γ appears in q and $\gamma < \delta$ generate W over V, where $\delta = v(q')$. Take $S =$ the set of elements $\gamma \in \overline{V}$ such that there exists $a \in \overline{F}$ such that $v(q + q' - a) = \gamma$ and let ϕ be obtained from $q + q'$ by truncating the terms with value not in S. Then $F' = F(\phi)$ is immediate transcendental, and clearly $V' = W$ by Theorem 1.1 and Theorem 2.4. Next suppose $V \subseteq W \subseteq \overline{V}$ and that W/V is countably infinite. Pick generators $\alpha_1, \alpha_2, ...$ for W over V. We can assume each α_i is positive. Let $\beta_1 = \alpha_1, \beta_2 = \alpha_1 + \alpha_2, ...$ Then $\beta_1 < \beta_2 < ...$ and these elements generate W over V. Pick $q_i \in \overline{F}$ so that the value group of $F(q_i)$ is contained in W and $v(q_i) = \beta_i$ (using Case 1). Let $p_1 = q_1$, $i_1 = 1$. Not all of the β_i are in the value group of $F(p_1)$. Let i_2 be the first i such that β_i is not in the value group of $F(p_1)$, and let $p_2 = p_1 + q_{i_2}$. Similarly, let i_3 be the first i such that β_i is not in the value group of $F(p_2)$ and let $p_3 = p_2 + q_{i_3}$, etc.. In this way we get a sequence of elements $p_1, p_2, ...$ in $\overline{F}$. Let S be the set of elements of $\overline{V}$ which are $< \beta_{i_k}$ for some k (so the sequence $\beta_{i_1} < \beta_{i_2} < ...$ is cofinal in S and S has no largest element). Also take ϕ to be the power series in $\overline{\kappa}((\overline{V}))$ such that ϕ has no term in x^γ if $\gamma \notin S$, and such that $v(\phi - p_j) \geq \beta_{i_j}$. Then $F' = F(\phi)$ is immediate transcendental over F and $V' = W$. $\square$

5.3 Remark. It can be shown that also the residue field extension can be prescribed arbitrarily, subject to the restriction imposed by Theorem 5.1; cf. [12].

6. Characterization of $\aleph_0$–saturated real closed fields.

Let an ordinal α be given. Recall that a totally ordered set Δ is an η_α–set, if given Δ_1, Δ_2 subsets of Δ of cardinality $< \aleph_\alpha$ and such that $\Delta_1 < \Delta_2$ then there exists $\delta \in \Delta$ such that $\Delta_1 < \delta < \Delta_2$. For example Δ is a dense linear orderings without endpoints if and only if Δ is an η_0–set. Also, since the theory of dense linear orderings without endpoints admits elimination of quantifiers [22], it follows that a dense linear ordering without endpoints is $\aleph_\alpha$–saturated (as ordered set) if and only if it is an η_α–set. In [1], a beautiful valuation theoretic characterization of ordered Abelian groups and ordered fields which are η_α–sets for a nonzero α is given. Divisible ordered Abelian groups (respectively real closed fields) are o–minimal [18]. Thus for $\aleph_\alpha > \aleph_0$, they are $\aleph_\alpha$–saturated as ordered groups (respectively ordered fields) if and only if they are $\aleph_\alpha$–saturated as ordered sets, that is, if and only if they are η_α–sets [13]. Thus one obtains easily from Alling's characterization a characterization of the $\aleph_\alpha$–saturated divisible ordered Abelian groups and real closed fields, for *uncountable* $\aleph_\alpha$. This characterization however does not cover the case of the $\aleph_0$–saturated models. In [13] this case was studied for divisible ordered Abelian groups:

6.1 Theorem. *Let $G \neq 0$ be a divisible ordered Abelian group. Then G is $\aleph_0$–saturated (in the language of ordered groups) if and only if its value set is a dense linear ordering without endpoints and all its archimedean components are isomorphic to $\mathbb{R}$.*

We consider here the case of real closed fields. Below, let R denote a real closed field, v its natural valuation, G its value group and ρ its residue field. We prove the following

6.2 Theorem. *Let $\aleph_\alpha \geq \aleph_0$, then R is $\aleph_\alpha$–saturated (in the language of ordered fields) if and only if*
1) G is $\aleph_\alpha$–saturated
2) $\rho \simeq \mathbb{R}$
3) every pseudo Cauchy sequence in a subfield of absolute transcendence degree less than $\aleph_\alpha$ has a pseudo limit in R.

Before proving the theorem let us note the following. Since R is real closed, G is divisible, and thus, by elimination of quantifiers [22], is o-minimal as noted above. More precisely, a formula $\phi(x, g_1, ..., g_n)$ in one free variable x and parameters $g_1, ..., g_n \in G$ is equivalent to a finite union of intervals with endpoints in $\langle g_1, \cdots, g_n\rangle \cup \{-\infty, +\infty\}$ (for $A \subset G$, $\langle A\rangle$ denotes the smallest divisible subgroup of G containing A). Thus if q is a complete 1-type over G with parameters in $A \subset G$, to realize q it is necessary and sufficient to realize the following set of formulas (determined by q):

$$\{g \leq x\,;\; g \in \langle A\rangle\;,\, q \vdash g \leq x\} \cup \{x \leq h\;;\; h \in \langle A\rangle\;,\, q \vdash x \leq h\}.$$

Similarly, by elimination of quantifiers [22], R is o-minimal. That is, a formula $\phi(x, a_1, \ldots, a_n)$ with parameters in $A = \{a_1, \ldots, a_n\} \subset R$, is equivalent to a finite union of intervals with endpoints in $\overline{\mathbb{Q}(A)} \cup \{-\infty, +\infty\}$ (here $\overline{\mathbb{Q}(A)}$ denotes the relative algebraic closure of $\mathbb{Q}(A)$ in R.) Thus if q is a complete 1-type over R with parameters in $A \subset R$, to realize q it is necessary and sufficient to realize the set:

$$\{b \leq x\,;\ b \in \overline{\mathbb{Q}(A)}\,,\ q \vdash b \leq x\} \cup \{x \leq c\,;\ c \in \overline{\mathbb{Q}(A)}\,,\ q \vdash x \leq c\}\ .$$

Now we prove the theorem. Suppose that 1), 2), 3) are satisfied, we show that R is $\aleph_\alpha$–saturated. By 2), we identify ρ with $\mathbb{R}$.

Let $A \subset R$ with $\operatorname{card} A < \aleph_\alpha$ and q a 1-type over R with parameters in A (w.l.o.g., assume that q is a complete type). We want to realize q in R. By the remark preceeding the proof, it is enough to realize in R the set

$$q'(x) = \{b \leq x\,;\ b \in \overline{\mathbb{Q}(A)}\,,\ q \vdash b \leq x\} \cup \{x \leq c\,;\ c \in \overline{\mathbb{Q}(A)}\,,\ q \vdash x \leq c\}\ .$$

If $q'(x)$ contains an equality, the result is obvious. So suppose that we only have strict inequalities in $q'(x)$.

Set $R' = \overline{\mathbb{Q}(A)}$, $B = \{b \in R'\,;\ q \vdash b < x\}$, $C = \{c \in R'\,;\ q \vdash x < c\}$. Let R'' be an elementary extension of R in which q is realized; and $x_0 \in R''$ s.t. $R'' \models q(x_0)$. Consider the following subset of $v(R'')$:

$$\Delta = \{v(d - x_0)\,;\ d \in R'\}\ .$$

There are three cases to consider.

Immediate transcendental case: Δ has no largest element. Thus,

$$\forall d \in R'\ \exists d' \in R' :\ v(d' - x_0) > v(d - x_0).$$

Let $\{v(d_\lambda - x_0)\}_{\lambda<\mu}$ be cofinal in Δ, then $\{d_\lambda\}_{\lambda<\mu}$ is a pseudo Cauchy sequence in R' and $\operatorname{trdeg}(R'|\mathbb{Q}) < \aleph_\alpha$. Let $a \in R$ be a pseudo limit. We claim that a realizes $q'(x)$; indeed we have:

$$v(a - x_0) = v(a - d_\lambda + d_\lambda - x_0) \geq \min\{v(a - d_\lambda), v(d_\lambda - x_0)\}$$

and

$$v(a - d_\lambda) = v(d_{\lambda+1} - d_\lambda) = v(x_0 - d_\lambda)\ ,$$

thus for all λ we have $v(a - x_0) \geq v(d_\lambda - x_0)$; thus for all $d \in R'$ we have $v(a - x_0) > v(d - x_0)$.

Let $b \in B$; we show that $b < a$. If not, $b \geq a$, so $a \leq b < x_0$, which implies that $v(b - x_0) \geq v(a - x_0)$, contradiction. Similarly one shows that if $c \in C$, then $a < c$.

Value transcendental case: Assume now that Δ has a largest element $\gamma \notin v(R')$, and fix $d_0 \in R'$ s.t. $v(d_0 - x_0) = \gamma$ is the maximum of Δ. Note that if $d_0 \in B$ then for all $d \in R'$ with $v(d - x_0) = \gamma$ we must have that $d \in B$, and if $d_0 \in C$ then for all $d \in R'$ with $v(d - x_0) = \gamma$ we must have that $d \in C$. Assume first that $d_0 \in B$ (the case $d_0 \in C$ is treated similarly). Consider $\Delta_1 = \{v(c - d_0)\,;\; c \in C\}$ and $\Delta_2 = \{v(b - d_0)\,;\; b \in B, b > d_0\}$, we claim that $\Delta_1 < \gamma < \Delta_2$. Indeed since $d_0 \in B$, we must have $v(c - x_0) < \gamma$ for all $c \in C$, thus

$$v(c - d_0) = v(c - x_0 + x_0 - d_0) = \min\{v(c - x_0), v(x_0 - d_0)\} = v(c - x_0) < \gamma.$$

If $b \in B$, $b \geq d_0$ then $v(x_0 - b) \geq v(x_0 - d_0) = \gamma$ and equality must hold since γ is the largest element. Thus,

$$v(b - d_0) = v(b - x_0 + x_0 - d_0) \geq \min\{v(b - x_0), v(x_0 - d_0)\} = \gamma.$$

Since $\gamma \notin v(R')$ we must have $v(b - d_0) > \gamma$. This establishes our claim. Hence,

$$t(y) = \{v(c - d_0) < y\,;\; c \in C\} \cup \{y < v(b - d_0)\,;\; b \in B\,,\, b > d_0\}$$

is a type in G with parameters in $G' = v(R')$. If $\aleph_\alpha > \aleph_0$ then $\operatorname{card} G' < \aleph_\alpha$ and we can realize $t(y)$ in G by condition 1) of the theorem. If $\aleph_\alpha = \aleph_0$, then R' is a real closed field of finite absolute transcendence degree, so G' has finite rational rank [24; § 10]. Thus if $\{g_1, \dots, g_n\}$ is a $\mathbb{Q}$–basis of G', then $t(y)$ is in fact a type in the parameters $\{g_1, \dots, g_n\}$, hence can be realized by some $g \in G$ (since G is $\aleph_0$–saturated). Let $a \in R$, $a > 0$ s.t. $v(a) = g$, then (by definition of $t(y)$)

$$\forall c \in C\, \forall b \in B :\; b > d_0 \Longrightarrow v(c - d_0) < v(a) < v(b - d_0)\ ,$$

hence (by compatibility of v with the order)

$$\forall c \in C\, \forall b \in B :\; b > d_0 \Longrightarrow b - d_0 < a < c - d_0\ ,$$

i.e.,

$$\forall c \in C\, \forall b \in B :\; b > d_0 \Longrightarrow b < a + d_0 < c\ ,$$

which implies

$$\forall c \in C\, \forall b \in B: \; b < a + d_0 < c \,,$$

so $a + d_0 \in R$ realizes $q'(x)$ as required.

Residue transcendental case: Assume now that the largest element γ of Δ is in $v(R')$, and fix $a > 0$, $a \in R'$ and $d_0 \in R'$ s.t. $v(d_0 - x_0) = \gamma = v(a)$.
Claim: there exist $b_0 \in B$ and $c_0 \in C$ s.t. for all $b \in B$ with $b \geq b_0$ and for all $c \in C$ with $c \leq c_0$ we have

$$v(b - d_0) = \gamma = v(a) = v(c - d_0)$$

(note that then for such b's and c's we will have also that $v(b-x_0) = v(x_0-d_0) = \gamma = v(c - x_0)$).
Indeed, since $v(d_0 - x_0) = v(a)$ there exists $n \in \mathbb{N}$ s.t. $na > |x_0 - d_0| > a/n$. If $d_0 \in B$ (i.e. $d_0 < x_0$) we set $c_0 = d_0 + na$ and $b_0 = d_0 + a/n$. Clearly $c_0 > x_0$ so $c_0 \in C$ and $b_0 < x_0$ so $b_0 \in B$. Moreover $v(b_0 - d_0) = v(a/n) = v(a) = v(na) = v(c_0 - d_0)$. For $b \in B$, $b > b_0$ and $c \in C$, $c < c_0$ we have $d_0 < b_0 < b < c < c_0$. Thus $v(b - d_0) \leq v(b_0 - d_0) = \gamma$ and $\gamma = v(c_0 - d_0) \leq v(b - d_0)$ hence equality must hold. Similarly one shows that $\gamma = v(c - d_0)$.
If $d_0 \in C$ (i.e. $d_0 > x_0$) we set $b_0 = d_0 - na$ and $c_0 = d_0 - a/n$. Clearly $c_0 > x_0$ so $c_0 \in C$ and $b_0 < x_0$ so $b_0 \in B$. Moreover $v(b_0 - d_0) = v(-na) = v(a) = v(-a/n) = v(c_0 - d_0)$. For $b \in B$, $b > b_0$ and $c \in C$, $c < c_0$ we have $b_0 < b < c < c_0 < d_0$. Thus $\gamma = v(b_0 - d_0) \leq v(b - d_0)$ and $\gamma = v(c_0 - d_0) \geq v(b - d_0)$, thus equality must hold. Similarly one shows that $\gamma = v(c - d_0)$. This establishes the claim.

Consider

$$\left\{ \frac{b - d_0}{a} < x'' \,;\; b \in B \,,\, b \geq b_0 \right\} \cup \left\{ x'' < \frac{c - d_0}{a} \,;\; c \in C \,,\, c \leq c_0 \right\} \,.$$

If $x'' \in R$ realizes this, then $x' = x'' \cdot a \in R$ realizes

$$\{ b - d_0 < x' \,;\; b \in B \,,\, b \geq b_0 \} \cup \{ x' < c - d_0 \,;\; c \in C \,,\, c \leq c_0 \}$$

thus $x' + d_0 \in R$ realizes $q'(x)$. Thus we just need to find x''. Assume that $d_0 \in B$ (the case $d_0 \in C$ is treated similarly). Note that by the claim for all $b \in B$ with $b \geq b_0$ and for all $c \in C$ with $c \leq c_0$ we have

$$v\left(\frac{b - d_0}{a} \right) = v\left(\frac{x_0 - d_0}{a} \right) = v\left(\frac{c - d_0}{a} \right) = 0 \,,$$

and taking residues,

$$\overline{\frac{b-d_0}{a}} < \overline{\frac{x_0-d_0}{a}} < \overline{\frac{c-d_0}{a}}$$

(note that the inequalities are strict, since otherwise we get a contradiction to the maximality of $v(a)$ in Δ). Consider the cut in the reals

$$\left\{\overline{\frac{b-d_0}{a}}\,;\ b \in B\,,\, b \geq b_0\right\} \cup \left\{\overline{\frac{c-d_0}{a}}\,;\ c \in C\,,\, c \leq c_0\right\}\ .$$

By the above strict inequalities, $r'' := \overline{\frac{x_0-d_0}{a}} =\in \mathbb{R}$ satisfies that

$$\left\{\overline{\frac{b-d_0}{a}}\,;\ b \in B\,,\, b \geq b_0\right\} < r'' < \left\{\overline{\frac{c-d_0}{a}}\,;\ c \in C\,,\, c \leq c_0\right\}\ .$$

Let $x'' \in R$ s.t. $\overline{x''} = r''$ (x'' exists by condition 2) of the theorem). This completes the residue transcendental case, and the only if direction of the proof.

Now suppose that R is $\aleph_\alpha$–saturated. We show that 1), 2), 3) must hold.

1): Let $q(x)$ be a type in the parameters $\{g_\alpha\,;\ \alpha \in \lambda\} \subset G$ with $\lambda < \aleph_\alpha$ (w.l.o.g. assume q is a complete type). Set $H = \langle\{g_\alpha\,;\ \alpha \in \lambda\}\rangle$. It suffices to realize in G the set

$$\{g \leq x\,;\ g \in H\,,\, q(x) \vdash g \leq x\} \cup \{x \leq g\,;\ g \in H\,,\, q(x) \vdash x \leq g\}\ .$$

If the set contains an equality, the result is clear. So suppose that we only have strict inequalities.

For every $\alpha \in \lambda$, fix $a_\alpha \in R$, $a_\alpha > 0$ s.t. $v(a_\alpha) = g_\alpha$. If $g \in H$ and $g = q_1 g_{\alpha_1} + \ldots + q_n g_{\alpha_n}$ with $q_1, \ldots, q_n \in \mathbb{Q}$, then $g = v(a_{\alpha_1}^{q_1} \cdot \ldots \cdot a_{\alpha_n}^{q_n})$ where for simplicity we agree to choose $a_{\alpha_i}^{q_i} > 0$ for all $i \in \{1, \ldots, n\}$. Let

$$H_1 = \{g \in H\,;\ q(x) \vdash g < x\} \text{ and } H_2 = \{g \in H\,;\ q(x) \vdash x < g\}$$

and consider

$$\begin{aligned} q'(x) = \{&n a_{\alpha_1}^{q_1} \cdot \ldots \cdot a_{\alpha_n}^{q_n} < x\,;\ n \in \mathbb{N},\, v(a_{\alpha_1}^{q_1} \cdot \ldots \cdot a_{\alpha_n}^{q_n}) \in H_2\} \\ &\cup \{mx < a_{\alpha_1}^{q_1} \cdot \ldots \cdot a_{\alpha_n}^{q_n}\,;\ m \in \mathbb{N},\, v(a_{\alpha_1}^{q_1} \cdot \ldots \cdot a_{\alpha_n}^{q_n}) \in H_1\}\ . \end{aligned}$$

Since R is a dense linear ordering without endpoints, $q'(x)$ is finitely realizable in R. Thus $q'(x)$ is a type in the parameters $\{a_\alpha\,;\ \alpha \in \lambda\}$ with $\lambda < \aleph_\alpha$. Hence $q'(x)$ is realized in R, say by a. It is clear that $v(a)$ realizes $q(x)$.

2): By hypothesis, $(R, +, 0, <)$ is $\aleph_0$–saturated, so by Theorem 3.1, all archimedean components are isomorphic to $\mathbb{R}$. In particular, $\rho \simeq \mathbb{R}$.

3): Let $(a_\nu)_{\nu<\mu}$ be a pseudo Cauchy sequence in a subfield R_0 of R s.t. $\operatorname{trdeg}(R_0|\mathbb{Q}) = \lambda < \aleph_\alpha$. Let $\{b_\alpha\,;\ \alpha \in \lambda\}$ be a transcendence basis of R_0 over $\mathbb{Q}$. Then every element of R_0 is definable by a formula with parameters in $\{b_\alpha\,;\ \alpha \in \lambda\}$. Consider

$$q_1(x) = \{n|x - a_{\nu+1}| < |a_\nu - a_{\nu+1}|\,;\ \nu < \mu\,,\, n \in \mathbb{N}\}\ ;$$

then $q_1(x)$ is a set of formulas in the parameters $\{b_\alpha\,;\ \alpha \in \lambda\}$. Moreover, $q_1(x)$ is finitely satisfied in R since $(a_\nu)_{\nu<\mu}$ is pseudo Cauchy. Hence $q_1(x)$ is a type, and it is clear that a realization of $q_1(x)$ in R is a pseudo limit of the sequence. This completes the proof of the theorem.

6.3 Corollary. *If R is a maximally valued real closed field, then R is not saturated.*

Proof. Let $\operatorname{card} R = \aleph_\alpha$, R saturated and maximally valued. Let $G = v(R)$, then $R \simeq \mathbb{R}((G))$. By the theorem, G is also $\aleph_\alpha$–saturated and thus contains a well ordered subset of cardinality $\aleph_\alpha$. It follows that $\operatorname{card}\mathbb{R}((G)) \geq 2^{\aleph_\alpha} > \aleph_\alpha$, contradiction. □

References

1. N.L. Alling, *On the existence of real closed fields that are η_α–sets of power $\aleph_\alpha$*, Trans. Amer. Math. Soc. **103** (1962), 341–352.
2. M.E. Alonso, *Ordenes y clausura semientera*, PhD. Thesis, Univ. Complutense de Madrid, 1984.
3. M.E. Alonso, J.M. Gamboa, J.M. Ruiz, *On orderings in real surfaces*, J. Pure Appl. Algebra **36** (1985), 1–14.
4. C. Andradas, L. Bröcker, J.M. Ruiz, *Constructible sets in real geometry*, Ergeb. Math., vol. 33, Springer-Verlag, 1996.
5. J. Bochnak, M. Coste, M-F. Roy, *Géométrie Algébrique Réelle*, Ergeb. Math., Springer-Verlag, 1987.
6. L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, New York, 1963.
7. R. Gilmer, *Extension of an order to a simple transcendental extension*, in Ordered fields and real algebraic geometry (San Francisco, 1981), eds. D. Dubois and T. Récio, Cont. Math., vol. 8, Amer. Math. Soc., 1982, pp. 113–118.

8. H. Hahn, *Über die nichtarchimedischen Grössensysteme*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Vienna, Section IIa, **116** (1907), 601–653.
9. I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. **9** (1942), 303–321.
10. M. Knebusch, C. Scheiderer, *Einführung in reelle Algebra*, Aufbaukurs Math., Vieweg, Braunschweig Wiesbaden, 1989.
11. W. Krull, *Allgemeine Bewertungstheorie*, J. für Math. **167** (1932), 160–196.
12. F.-V. Kuhlmann, *Valuations on Rational Function Fields, I: Implicit Constant Fields*, in preparation.
13. S. Kuhlmann, *Groupes abéliens divisibles ordonnés*, Publications mathématiques de l'Université Paris VII, Séminaire sur les Structures Algébriques Ordonnées, Sélection d'exposés 1984–1987, Vol. **1** (1990), 3–14.
14. S. MacLane, O.F.G. Schilling, *Zero dimensional branches of rank one on algebraic varieties*, Annals of Math. **40** (1939), 507–520.
15. M. Marshall, *Spaces of orderings and abstract real spectra*, Lecture Notes in Math., vol. 1636, Springer, 1996.
16. M.H. Mourgues, J.P. Ressayre, *Every real closed field has an integer part*, J. Symb. Logic **58** (1993), 641–647.
17. J. Ohm, *The Ruled Residue Theorem for simple transcendental extensions of valued fields*, Proc. Amer. Math. Soc. **89** (1983), 16–18.
18. A. Pillay, C. Steinhorn, *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. **295** (1986), 565–592.
19. A. Prestel, *Lectures on formally real fields*, IMPA, Rio de Janeiro, 1974; Lecture Notes in Math., vol. 1093, Springer, 1984.
20. S. Prieß-Crampe, *Angeordnete Strukturen. Gruppen, Körper, projektive Ebenen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 98, Springer, 1983.
21. P. Ribenboim, *Théorie des valuations*, Les Presses de L'Université de Montreal, 1968.
22. A. Robinson, *Complete Theories*, Amsterdam, 1956.
23. R. Rolland, *Étude des coupures dans les groupes et corps ordonnés*, in Géométrie algébrique réelle et formes quadratiques (Rennes, 1981), eds. J-L. Colliot-Thélène, M. Coste, L. Mahé, M-F. Roy, Lecture Notes in Mathematics, vol. 959, Springer, 1982, pp. 386–405.
24. P. Samuel, O. Zariski, *Commutative Algebra, Vol. II*, New York–Heidelberg–Berlin, 1960.
25. O.F.G. Schilling, *The theory of valuations*, Math. Surveys, vol. 4, Amer. Math. Soc., 1950.
26. M. Zekavat, *Orderings, cuts and formal power series*, PhD. Thesis, Univ. of Saskatchewan, in preparation.

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