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Valuation theory of exponential Hardy fields I*

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Dedicated to Murray Marshall on the occasion of his 60th birthday

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1 Introduction

In this paper, we analyze the structure of the Hardy fields associated with o-minimal expansions of the reals with exponential function. In fact, we work in the following more general setting. We take T to be the theory of a polynomially bounded o-minimal expansion $\mathcal{P}$ of the ordered field of real numbers. By $\mathcal{F}_T$ we denote the set of all 0-definable functions of $\mathcal{P}$. Further, we assume that T defines the restricted exponential and logarithmic functions (cf. [D–M–M1]). Then also $T(\exp)$ is o-minimal (cf. [D–S2]). Here, $T(\exp)$ denotes the theory of the expansion $(\mathcal{P}, \exp)$ where exp is the un-restricted real exponential function. Finally, we take any model $\mathcal{R}$ of $T(\exp)$ which contains $(\mathbb{R}, +, \cdot, <, \mathcal{F}_T, \exp)$ as a substructure. Then we consider the Hardy field $H(\mathcal{R})$ (see Section 2.2 for the definition) as a field equipped with convex valuations. Theorem B of [D–S2] tells us that $T(\exp)$ admits quantifier elimination and a universal axiomatization in the language augmented by a symbol for the inverse function log of exp. This implies that $H(\mathcal{R})$ is equal to the closure of its subfield $\mathcal{R}(x)$ under $\mathcal{F}_T$, exp and log; here, x denotes the germ of the identity function (cf. [D–M–M1], §5; the arguments also hold in the case where $\mathcal{R}$ is a non-archimedean model).

We shall analyze the valuation theoretical structure of this closure by explicitly showing how it can be built up from $\mathcal{R}(x)$ (cf. Section 3.3). Our construction method yields the following result (see Section 3.4 for definitions):

Theorem 1.1 *Every model $\mathcal{R}$ as chosen above is levelled.*

This implies that $T(\exp)$ *has levels with parameters*, in the sense of [M–M], and is exponentially bounded (cf. Theorem 3.11). We can determine the level of a function explicitly: it is the difference of two numbers which come up naturally in our construction method.

In Section 3.5 we use our main structure theorem (Theorem 3.11) to deduce:

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Theorem 1.2 *Suppose that for all $r \in \mathbb{R}$, $\mathcal{F}_T$ contains the power function*

$$\begin{array}{rccc} P_r : & (0,\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^r . \end{array}$$

Let $\mathcal{R}_T$ denote the reduct of $\mathcal{R}$ to the language of T. Then the Hardy field $H(\mathcal{R}_T)$ is maximal among the Hardy subfields of $H(\mathcal{R})$ associated with polynomially bounded reducts of $\mathcal{R}$.

L. v. d. Dries conjectured that

$$\mathbb{R}_{\text{an,powers}} = (\mathbb{R},+,\cdot,0,1,<,\mathcal{F}_{\text{an}},\{P_r \mid r \in \mathbb{R}\}) ,$$

the expansion of the ordered field of real numbers by the set $\mathcal{F}_{\text{an}}$ of restricted analytic functions and the power functions P_r , is a *maximal* polynomially bounded reduct of

$$\mathbb{R}_{\text{an,exp}} = (\mathbb{R},+,\cdot,0,1,<,\mathcal{F}_{\text{an}},\exp) ,$$

At least on the level of Hardy fields, this is true: since the elementary theory of $\mathbb{R}_{\text{an,powers}}$ is polynomially bounded and o-minimal and the power functions are definable in $\mathbb{R}_{\text{an,exp}}$ (cf. [M]), the foregoing theorem shows (cf. Theorem 3.16 for a more general result):
$H(\mathbb{R}_{\text{an,powers}})$ *is maximal among the Hardy subfields of* $H(\mathbb{R}_{\text{an,exp}})$ *associated with polynomially bounded reducts of* $\mathbb{R}_{\text{an,exp}}$.

In a subsequent paper, we shall study the residue fields of exponential Hardy fields with respect to arbitrary convex valuations (which are not necessarily $T(\exp)$-convex).

2 Some preliminaries

If (K,w) is a valued field, then we write wa for the value of $a \in K$ and wK for its value group $\{wa \mid 0 \neq a \in K\}$. Further, we write aw for the residue of a, and Kw for the residue field. The valuation ring is denoted by $\mathcal{O}_w$. For generalities on valuation theory, see [R]. For the general notions and tools from model theory we use in this paper, we refer the reader to [C–K].

2.1 Convex valuations

A valuation w on an ordered field K is called **convex** if $\mathcal{O}_w$ is convex. The convex valuation rings of an ordered field are linearly ordered by inclusion. If $\mathcal{O}_w \subsetneq \mathcal{O}_{w'}$ then w is said to be **finer** than w'. There is always a finest convex valuation, called the **natural valuation**. It is characterized by the fact that its residue field is archimedean. A valuation w on an ordered field is convex if and only if the natural valuation is finer or equal to w. **Throughout this paper, v will always denote the natural valuation,** unless stated otherwise.

If a, b are elements of an ordered group or an ordered field, then we write $a \ll b < 0$ if $a < b < 0$ and $\forall n \in \mathbb{N} : a < nb$. Similarly, $a \gg b > 0$ if $a > b > 0$ and $\forall n \in \mathbb{N} : a > nb$. We set $|a| := \max\{a, -a\}$. Then the natural valuation is characterized by:

$$va < vb \;\Leftrightarrow\; |a| \gg |b| \;. \tag{1}$$

Note that if $\mathbb{R} \subset K$ and $a \in K$ with $va = 0$, then there is some $r \in \mathbb{R}$ such that $v(a - r) > 0$. Further, $wr = 0$ for every $r \in \mathbb{R}$ and every convex valuation w.

Lemma 2.1 *Let v, w be arbitrary valuations on some field K. Suppose that v is finer than w. Then for all $a, b \in K$,*

$$va \le vb \;\Rightarrow\; wa \le wb \;. \tag{2}$$

In particular, $wa > 0 \Rightarrow va > 0$. Further, $H_w := \{vz \mid z \in K \wedge wz = 0\}$ is a convex subgroup of the value group vK of v. We have that $vz \in H_w \Leftrightarrow z \in \mathcal{O}_w^\times$. There is a canonical isomorphism $wK \simeq vK/H_w$. Conversely, every convex subgroup of vK is of the form H_w for some valuation w such that v finer or equal to w.

The valuation v of K induces a valuation v/w on Kw. There are canonical isomorphisms $v/w(Kw) \simeq H_w$ and $(Kw)v/w \simeq Kv$. If Kw is embedded in $\mathcal{O}_w$ such that the restriction of the residue map is the identity on Kw, then $v/w = v|_{Kw}$ (up to equivalence). Writing v instead of $v|_{Kw}$, we then have that $v(Kw) = H_w$ and $(Kw)v = Kv$.

We will call H_w the **convex subgroup associated with** w and w the **valuation associated with** H_w. Since the isomorphism is canonical, we will write $wK = vK/H_w$.

The order type of the chain of nontrivial convex subgroups of an ordered abelian group G is called the **rank** of G. If finite, then the rank is not bigger than the maximal number of rationally independent elements in G. In particular, G has finite rank if it is finitely generated or equivalently, if its divisible hull is a $\mathbb{Q}$-vector space of finite dimension.

From (1) and (2) it follows that for every convex valuation w,

$$|a| \le |b| \;\Rightarrow\; wa \ge wb \;. \tag{3}$$

For the rest of this section, we will assume that $(M, \exp)$ is a model of the elementary theory of $(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ such that $\mathbb{R} \subset M$ and the restriction of $\exp$ to $\mathbb{R}$ is the natural exponential $\exp$ on $\mathbb{R}$. Further, we take w to be any convex valuation on M. Then the exponential exp of M is an order preserving isomorphism from the additive group of M onto its multiplicative group of positive elements. Its inverse is the logarithm log; it is order preserving and defined for all positive elements. Consequently, if $z \in M$ is positive infinite, that is, $z > \mathbb{R}$, then $\log z > \log(\{r \in \mathbb{R} \mid r > 0\}) = \mathbb{R}$. In other words,

$$vz < 0 \wedge z > 0 \;\Rightarrow\; v\log z < 0 \wedge \log z > 0 \;. \tag{4}$$

Further, exp satisfies the Taylor axiom scheme:

(TA) $\qquad |z| \le 1 \;\Rightarrow\; |\exp z - \sum_{n=0}^{m} \frac{z^n}{n!}| < |z^m| \qquad (m \in \mathbb{N})$.

In order to derive a valuation theoretical property from this axiom, we need the following simple lemma:

Lemma 2.2 *Let K be an ordered field and w a convex valuation on K. Suppose that $h \in K$ satisfies*

$$\left| h - \sum_{k=0}^{m} s_k z_k \right| < |s'_m z_m| \quad \textit{for all } m \in \mathbb{N}, \tag{5}$$

where $s_k, s'_k \in \mathbb{R} \setminus \{0\}$, and $z_k \in K$ are such that $wz_{k+1} > wz_k$. Write

$$S_m := \sum_{k=0}^{m} s_k z_k .$$

Then $(S_m)_{m\in\mathbb{N}}$ is a pseudo Cauchy sequence in (K, w). Further,

$$w(h - S_m) = wz_{m+1} = w(S_{m+1} - S_m) , \tag{6}$$

which shows that h is a limit of this sequence.

Proof: Recall that $ws = 0$ for $0 \neq s \in \mathbb{R}$, and that $w|a| = wa$ for every a in K. By (5) and (3), we have that

$$\begin{aligned} w(h - S_m - s_{m+1}z_{m+1} - s_{m+2}z_{m+2}) &= w(h - S_{m+2}) \geq ws'_{m+2}z_{m+2} = wz_{m+2} \\ &> wz_{m+1} = ws_{m+1}z_{m+1} . \end{aligned}$$

By the ultrametric triangle law,

$$w(s_{m+1}z_{m+1} + s_{m+2}z_{m+2}) = \min\{ws_{m+1}z_{m+1}, ws_{m+2}z_{m+2}\} = ws_{m+1}z_{m+1} .$$

Hence, again by the ultrametric triangle law,

$$\begin{aligned} w(h - S_m) &= \min\{w(h - S_m - s_{m+1}z_{m+1} - s_{m+2}z_{m+2}), w(s_{m+1}z_{m+1} + s_{m+2}z_{m+2})\} \\ &= ws_{m+1}z_{m+1} = w(S_{m+1} - S_m) . \end{aligned}$$

□

Lemma 2.3 *For every $z \in M$,*

$$wz > 0 \;\Rightarrow\; w\exp z = 0 \wedge w(\exp z - 1) = wz \tag{7}$$

$$vz = 0 \;\Rightarrow\; v\exp z = 0 . \tag{8}$$

Proof: By Lemma 2.1, $wz > 0$ implies $vz > 0$, that is, z is infinitesimal. In particular, $|z| < 1$, and (TA) holds. Applying (6) of Lemma 2.2 with $m = 1$ and $z_m = z^m$, we find that $w(\exp z - 1 - z) = wz^2 = 2wz > wz$. By the ultrametric triangle law, this implies that $w\exp z = w(1 + z) = w1 = 0$ and $w(\exp z - 1) = wz$. This proves (7).

Now assume that $vz = 0$. Then there is some $r \in \mathbb{R} \subset M$ such that $v(z - r) > 0$. We have that $\exp r \in \mathbb{R}$, hence $v\exp r = 0$. By (7) with $w = v$, $v\exp(z - r) = 0$. Thus, $v\exp z = v\exp r\exp(z - r) = v\exp r + v\exp(z - r) = 0$. This proves (8). □

With M as before, exp also satisfies the following growth axiom scheme:

(GA) $\qquad z > m^2 \implies \exp z > z^m \qquad\qquad (m \in \mathbb{N})\,.$

From this, we derive:

Lemma 2.4 *For every $z \in M$,*

$$\begin{aligned} wz < 0 \,\wedge\, z > 0 \;&\Rightarrow\; w\exp z \ll wz \ll w\log z < 0 & (9)\\ wz = 0 \,\wedge\, z > 0 \;&\Rightarrow\; w\log z \geq 0 & (10)\\ vz \geq 0 \;&\Leftrightarrow\; v\exp z = 0\,. & (11)\end{aligned}$$

Proof: If $wz < 0$ and $z > 0$, then $z > \mathbb{R}$ and thus, $z > m^2$ for every $m \in \mathbb{N}$. So by (GA), $\exp z > z^m > 0$ for all m. Hence by (3), $w\exp z \leq mwz$ for all m, i.e., $w\exp z \ll wz < 0$. In view of (4), we can replace z by $\log z$ to get that $wz \ll w\log z < 0$. This proves (9).

Now assume that $wz = 0$ and $z > 0$. If $vz < 0$, then by (9), $vz < v\log z < 0$. If $vz > 0$, then $vz^{-1} < 0$ and by (9), $vz^{-1} < v\log z^{-1} = v(-\log z) = v\log z < 0$. In both cases, it follows from Lemma 2.1 that $0 = wz = wz^{-1} \leq w\log z \leq 0$, i.e., $w\log z = 0$. Now let $vz = 0$. If $v\log z < 0$, then by (9), $vz = v\exp\log z < 0$ if $\log z > 0$, and $vz = -vz^{-1} = -v\exp(-\log z) > 0$ if $\log z < 0$. Hence, $v\log z \geq 0$, and again by Lemma 2.1, $w\log z \geq 0$. This proves (10).

Implication "⇒" of (11) follows from (7) with $w = v$, together with (8). The converse implication follows from (10), where we take $w = v$ and replace z by $\exp z$. □

For positive infinite elements $z \in M$ and $m \in \mathbb{Z}$, we set $\log_0 z = z$, $\log_{m+1} z = \log(\log_m z)$ if $m \geq 0$, and and $\log_{m-1} z = \exp(\log_m z)$ if $m \leq 0$; note that every $\log_m z$ is again positive infinite. Similarly, we define $\exp_m z$ for every $z \in M$.

Corollary 2.5 *Assume that $\mathcal{R}$ is an exp-closed subfield of M. If $x \in M$ such that $wx < w\mathcal{R}$ and $x > 0$, then for $m > 1$,*

$$wx \ll w\log x \ll \ldots \ll w\log_m x \ll \ldots < w\mathcal{R}\,. \qquad (12)$$

Proof: The part "$wx \ll w\log x \ll \ldots \ll w\log_m x$" follows from (9) by induction on m. Now suppose that there is a positive integer m and some $\alpha \in w\mathcal{R}$ such that $\alpha \leq w\log_m x$. Replacing α by $2\alpha \in w\mathcal{R}$ if necessary, we may assume that $\alpha < w\log_m x$. Take a positive element $a \in \mathcal{R}$ such that $wa = \alpha$. Then by virtue of (3), $0 < \log_m x < a$. It follows that $x < \exp_m a$, which implies that $wx \geq w\exp_m a \in w\mathcal{R}$. This proves that if $wx < w\mathcal{R}$ then $w\log_m x < w\mathcal{R}$ for all m. □

For further details on the valuation theory of exponential fields, see [KS2], [KS1] and [K–K1].

2.2 Hardy fields

Let us recall some basic facts about Hardy fields. Initially, they were only defined as fields consisting of germs at ∞ of real-valued functions. But we will work with a more general definition that has also been used by other authors lately. Assume that T is the theory of any o-minimal expansion of the ordered field of real numbers by real-valued functions, and that $\mathcal{R}$ is a model of T. The Hardy field of $\mathcal{R}$, denoted by $H(\mathcal{R})$, is the set of germs at ∞ of unary $\mathcal{R}$-definable functions $f : \mathcal{R} \to \mathcal{R}$. Then $H(\mathcal{R})$ is an ordered differential field which contains $\mathcal{R}$. Let $x \in H(\mathcal{R})$ be the germ of the identity function. Then $H(\mathcal{R})$ is the closure of $\mathcal{R}(x)$ under all 0-definable functions of $\mathcal{R}$.

By $v_{\mathcal{R}}$ we will denote the finest convex valuation on $H(\mathcal{R})$ which is trivial on $\mathcal{R}$. Then $v_{\mathcal{R}}a < 0$ if and only if $a > \mathcal{R}$. If f, g are non-zero unary $\mathcal{R}$-definable functions on $\mathcal{R}$, then we will denote their germs in $H(\mathcal{R})$ by the same letters. With this convention, the following holds:

$$v_{\mathcal{R}}f \;=\; v_{\mathcal{R}}g \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} \text{ is a non-zero constant in } \mathcal{R}\,. \tag{13}$$

(Note that "$x \to \infty$" means letting x outgrows every element of $\mathcal{R}$.) The functions f and g are **asymptotic on** $\mathcal{R}$ if and only if this constant is 1, and we have:

$$v_{\mathcal{R}}(f-g) \;>\; v_{\mathcal{R}}g \iff f \text{ and } g \text{ are asymptotic on } \mathcal{R}\,, \tag{14}$$

or in other words,

$$v\left(\frac{f}{g}-1\right) \;>\; v\mathcal{R} \iff f \text{ and } g \text{ are asymptotic on } \mathcal{R}\,, \tag{15}$$

3 Closures of $\mathcal{R}(x)$ under $\mathcal{F}$, log and exp

General assumptions: Throughout this section, we will assume that T is the theory of a polynomially bounded o-minimal expansion $\mathcal{P}$ of the ordered field of real numbers by real-valued functions. Further, we assume that T defines the restricted exp and log. Then also $T(\exp)$ is o-minimal (cf. [D–S2]). Here, $T(\exp)$ denotes the theory of the expansion $(\mathcal{P}, \exp)$ where exp is the un-restricted real exponential function.

The archimedean field

$$\mathrm{Q} \;:=\; \{r \in \mathbb{R} \mid \text{ the function } x \mapsto x^r :\; (0,\infty) \longrightarrow \mathbb{R} \text{ is 0-definable in } \mathcal{P}\}$$

is called the **field of exponents of** T.

We let $\mathcal{F}_T$ denote the set of function symbols in the language of T and assume that there is a function symbol in $\mathcal{F}_T$ for each 0-definable function of $\mathcal{P}$. This implies that T admits quantifier elimination and a universal axiomatization (cf. [D–L], §2). We let $\mathcal{F}$ denote any subset of $\mathcal{F}_T$.

Further, we assume that M is a model of T. (Later, we will assume that it is a model of $T(\exp)$, but we will not distinguish between this model and its reduct to the language of T.) Suppose that K is a submodel (hence an elementary substructure) of M. Take $x_i \in M$, $i \in I$. By $K\langle x_i \mid i \in I\rangle$ we denote the 0-definable closure of $K \cup \{x_i \mid i \in I\}$ in M. By our assumption on the language of T, this is the closure of $K \cup \{x_i \mid i \in I\}$ under $\mathcal{F}_T$, that is, the smallest subfield of M containing $K \cup \{x_i \mid i \in I\}$ and closed under all functions which interpret the function symbols of $\mathcal{F}_T$ in M. Since T admits a universal axiomatization and $K\langle x_i \mid i \in I\rangle$ is a substructure of M, it is a model of T. Since T admits quantifier elimination, $K\langle x_i \mid i \in I\rangle$ is an elementary substructure of M.

For an arbitrary subfield $F \subset M$, we let F^{Q} denote the smallest subfield of M which contains F and is Q-**closed**, i.e., closed under the exponents from Q. Further, we let $F^{\mathrm{rQ}\mathcal{F}}$ denote the smallest real closed subfield of M which contains F, is Q-closed, and is $\mathcal{F}$-**closed**, i.e., closed under all functions on M which are interpretations of function symbols in $\mathcal{F}$. We will say that F is rQ$\mathcal{F}$-**closed** if $F = F^{\mathrm{rQ}\mathcal{F}}$. Note that real closures can be taken to lie in M since M is real closed.

If F is Q-closed, then for every convex valuation w, the value group wF is a Q-vector space with scalar multiplication defined by $qw(a) = w(|a|^q)$ for $q \in \mathrm{Q}$. If $\alpha \in wF$, then $\mathrm{Q}\alpha$ shall denote the Q-subvector space generated by α. As Q always contains $\mathbb{Q}$, we see that wF^{Q} is always divisible.

3.1 Value groups

The following property (Lemma 3.1) of polynomially bounded o-minimal expansions of the reals was proved in full generality in [D] (Lemma 5.4); see also Corollary 3.7 of [D–M–M1]. Note that in the case of a polynomially bounded expansion, every convex valuation w of a model is T-convex (cf. [D–L], §4).

Lemma 3.1 *Assume that $\mathcal{R}$ is a submodel of M. If $x \in M$ such that $wx \notin w\mathcal{R}$, then $w\mathcal{R}\langle x\rangle = w\mathcal{R} \oplus \mathrm{Q}wx$.*

Lemma 3.2 *Assume that $\mathcal{R}$ is a submodel of M. Take elements $x_i \in M$, $i \in I$, such that the values wx_i, $i \in I$, are Q-linearly independent over $w\mathcal{R}$. Then*

$$w\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} = w\mathcal{R}(x_i \mid i \in I)^{\mathrm{Q}} = w\mathcal{R} \oplus \bigoplus_{i\in I} \mathrm{Q}wx_i\ . \tag{16}$$

Proof: Since every element of $\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}$ already lies in $\mathcal{R}(x_i \mid i \in I_0)^{\mathrm{rQ}\mathcal{F}}$ for a finite subset $I_0 \subseteq I$ and a similar assertion is true for the fields $\mathcal{R}(x_i \mid i \in I)^{\mathrm{Q}}$ and $\mathcal{R}\langle x_i \mid i \in I\rangle$, it suffices to prove our assertion for the case of I finite. We may write $I = \{1, \ldots, n\}$. By induction on n, Lemma 3.1 shows that

$$w\mathcal{R}\langle x_1, \ldots, x_n\rangle \;=\; w\mathcal{R} \oplus \bigoplus_{i=1}^{n} \mathrm{Q}wx_i\ . \tag{17}$$

Since $\mathcal{R}\langle x_1, \ldots, x_n\rangle$ is rQ$\mathcal{F}$-closed, we have that

$$\mathcal{R}(x_1, \ldots, x_n)^{\mathrm{Q}} \;\subseteq\; \mathcal{R}(x_1, \ldots, x_n)^{\mathrm{rQ}\mathcal{F}} \;\subseteq\; \mathcal{R}\langle x_1, \ldots, x_n\rangle\ .$$

As $w\mathcal{R}(x_1,\dots,x_n)^{\mathrm{Q}}$ is a Q-vector space and contains $wx_1,\dots,wx_n$, we obtain that

$$\begin{aligned} w\mathcal{R}\oplus\bigoplus_{i=1}^{n}\mathrm{Q}wx_i \;&\subseteq\; w\mathcal{R}(x_1,\dots,x_n)^{\mathrm{Q}} \;\subseteq\; w\mathcal{R}(x_1,\dots,x_n)^{\mathrm{rQ}\mathcal{F}} \\ &\subseteq\; w\mathcal{R}\langle x_1,\dots,x_n\rangle \;=\; w\mathcal{R}\oplus\bigoplus_{i=1}^{n}\mathrm{Q}wx_i\;, \end{aligned}$$

which shows that equality must hold everywhere. □

3.2 Linear independence of generating values

From now on, let M always be a model of $T(\exp)$, and $\mathcal{R}$ a submodel of M containing $(\mathbb{R},+,\cdot,<,\mathcal{F}_T,\exp)$ as a substructure. We take $\mathcal{F}$ as before, but always assume in addition that $\mathcal{F}$ contains function symbols for the restricted exp and log. Hence, if a subfield F of M is $\mathcal{F}$-closed, then $\exp\varepsilon\in F$ and $\log(1+\varepsilon)\in F$ for every infinitesimal ε in F. Since $\mathbb{R}\subseteq\mathcal{R}$, we have that $\mathcal{R}v=\mathbb{R}$.

Note that in view of Theorem B of [D–S2], $\mathcal{R}$ is an elementary substructure of M, and $(\mathbb{R},+,\cdot,<,\mathcal{F}_T,\exp)$ is an elementary substructure of both. However, we will not use this fact in our constructions.

For every subfield K of $\mathcal{O}_w$, its multiplicative group $K^\times$ is contained in the multiplicative group $\mathcal{O}_w^\times$ of all units of $\mathcal{O}_w$. We will say that K is **relatively exp-closed in** $\mathcal{O}_w^\times$ if $a\in K$ and $\exp(a)\in\mathcal{O}_w^\times$ implies that $\exp(a)\in K$. For example, $\mathbb{R}$ is relatively exp-closed in $\mathcal{O}_w^\times$ for every convex valuation w of M.

Lemma 3.3 *Let K be a log- and $\mathrm{rQ}\mathcal{F}$-closed subfield of M. Let w be a convex valuation of M. Assume that the residue field Kw is a subfield of $\mathcal{O}_w\cap K$, relatively exp-closed in $\mathcal{O}_w^\times$. Take any $a\in K$ such that $\exp a\notin K$. Then $w\exp a$ is* Q*-linearly independent over wK.*

Proof: Suppose that $w\exp a$ is not Q-linearly independent over wK. Since K is Q-closed, wK is a Q-vector space, and it follows that $w\exp a=wb\in wK$ for some positive $b\in K$. Then $w\frac{\exp a}{b}=0$ and by Lemma 2.4, $w(a-\log b)=w\log(\frac{\exp a}{b})\geq 0$. Since K is log-closed, $\log b\in K$. Hence, there is $c\in Kw$ such that $w(a-\log b-c)>0$. By Lemma 2.3, this shows that $w\,\frac{\exp a}{b\exp c}=w\exp(a-\log b-c)=0$. In particular, we find that $w\exp c=w\frac{\exp a}{b}=0$, that is, $\exp c\in\mathcal{O}_w^\times$. By assumption on Kw, $\exp c\in Kw\subset K$.

By Lemma 2.1, $w(a-\log b-c)>0$ yields that $v(a-\log b-c)>0$. Therefore, $\exp(a-\log b-c)\in K^{\mathcal{F}}=K$, showing that $\exp a=\exp(a-\log b-c)\cdot b\cdot\exp c\in K$. We conclude: if $\exp a\notin K$, then $w\exp a$ is Q-linearly independent over wK. □

Lemma 3.4 *Assume that $K=\mathcal{R}(x_i\mid i\in I)^{\mathrm{rQ}\mathcal{F}}\subset M$ such that*
1) the values vx_i, $i\in I$, are Q*-linearly independent over $v\mathcal{R}$,*
2) $x_i>0$ and $\log x_i\in K$ for all $i\in I$.
Then K is log-closed.

Proof: Take a positive $b \in K$. By virtue of Lemma 3.2, there is a finite subset $I_0 \subset I$ and $q_i \in \mathbb{Q}$ such that $vb = vr' + \sum_{i\in I_0} q_i v x_i$ for some positive $r' \in \mathcal{R}$. So we can write $b = r' \prod_{i\in I_0} x_i^{q_i} \cdot r \cdot (1+\varepsilon)$ with positive $r \in \mathbb{R}$ and some $\varepsilon \in K$ such that $v\varepsilon > 0$. We have that $\log(1+\varepsilon) \in K$ since K is $\mathcal{F}$-closed. Moreover, $\log r' \in \mathcal{R} \subset K$ and $\log r \in \mathbb{R} \subset K$. Therefore,

$$\log b \;=\; \log r' + \sum_{i\in I_0} q_i \log x_i \;+\; \log r \;+\; \log(1+\varepsilon) \;\in\; K\;.$$

□

Lemma 3.5 *Assume that K is of the form*

$$\left.\begin{array}{l} \mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} \;\; \textit{log-closed, with } x_i > 0 \textit{ and} \\ vx_i,\; i \in I,\; \mathbb{Q}\textit{-linearly independent over } v\mathcal{R}. \end{array}\right\} \tag{18}$$

Take any $a \in K$ such that $\exp a \notin K$. Then $v\exp a$ is $\mathbb{Q}$-linearly independent over vK,

$$vK(\exp a)^{\mathrm{rQ}\mathcal{F}} \;=\; vK \oplus \mathbb{Q}\, v \exp a\;. \tag{19}$$

Moreover, $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is again log-closed, and therefore of the form (18). It contains $\exp b$ whenever $b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ and $v\exp b$ is $\mathbb{Q}$-linearly dependent over $vK(\exp a)^{\mathrm{rQ}\mathcal{F}}$.

Proof: Applying Lemma 3.3 with $w = v$ and $Kw = \mathbb{R}$, we obtain that $v\exp a$ is $\mathbb{Q}$-linearly independent over vK and that $\exp b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ whenever $b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ and $v\exp b$ is $\mathbb{Q}$-linearly dependent over $vK(\exp a)^{\mathrm{rQ}\mathcal{F}}$. Equation (19) follows by an application of Lemma 3.2 to K and to $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$. Finally, we infer from Lemma 3.4 that $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is log-closed. □

Lemma 3.6 *Assume that $(\mathcal{R}, v) \subset (K, v)$ is any extension of valued fields and that w is a valuation on K such that v is finer than w, and $Kw = \mathcal{R}$. Take $x_i \in K$ such that the values vx_i, $i \in I$, are $\mathbb{Q}$-linearly independent over $v\mathcal{R}$. Then the values wx_i, $i \in I$, are $\mathbb{Q}$-linearly independent.*

Proof: From $Kw = \mathcal{R}$ it follows that v is the composition of w with the restriction of v to $\mathcal{R}$. Thus, $v\mathcal{R}$ is a convex subgroup of vK and there is a canonical isomorphism $wK \simeq vK/v\mathcal{R}$. Hence $\sum_{i\in I} q_i w x_i = 0$ (where $q_i \in \mathbb{Q}$, almost all of them zero) implies $\sum_{i\in I} q_i v x_i \in v\mathcal{R}$. By assumption, this implies that $q_i = 0$ for all $i \in I$. □

3.3 A basic construction

First, we show how to construct log-closed fields K as in (18). **From now on, we always assume that $x \in M$ such that $x > \mathcal{R}$, that is, $vx < v\mathcal{R}$ and $x > 0$. By $v_{\mathcal{R}}$ we will denote the finest convex valuation on M which is trivial on $\mathcal{R}$.**

Lemma 3.7 *The field*

$$\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$$

is log-closed. The convex hull of its value group in vM is equal to the smallest convex subgroup containing vx and $v\mathcal{R}$. If w is a convex valuation on M, trivial on $\mathcal{R}$ and such that $wx = 0$, then the field $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ lies in $\mathcal{O}_w$.

Proof: From Corollary 2.5 we know that

$$vx \ll v\log x \ll \ldots \ll v\log_m x \ll \ldots < v\mathcal{R}\,. \tag{20}$$

In particular, the values $v\log_m x$ lie in distinct archimedean classes. As Q is archimedean, it follows that the values $v\log_m x$ are Q-linearly independent over $v\mathcal{R}$. So it follows from Lemma 3.4 that $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ is log-closed.

From Lemma 3.2 we infer that $v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}} = v\mathcal{R} \oplus \bigoplus_{m\geq 0} \mathrm{Q}\, v\log_m x$. Now (20) yields that this group is contained in the smallest convex subgroup H of vM which contains vx and $v\mathcal{R}$. If w is as in our assumption, then H is contained in the convex subgroup H_w of vM associated with w. Thus, w is trivial on $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$, that is, this field lies in $\mathcal{O}_w$. □

For $\mathcal{F} \subseteq \mathcal{F}_T$ we denote by $LE_{\mathcal{R},\mathcal{F}}(x)$ the smallest subfield of M which contains $\mathcal{R}(x)$ and is real closed and closed under $\mathcal{F}$, exp **and** log. We shall show how to build up $LE_{\mathcal{R},\mathcal{F}}(x)$ from $\mathcal{R}(x)$. As a preparation for what we will need in a later paper, we will keep our construction more general. We will construct a variety of fields (described in Lemma 3.8 below) of which $LE_{\mathcal{R},\mathcal{F}}(x)$ is just a special case. Let w be a convex valuation on M, trivial on $\mathcal{R}$, and H_w its associated convex subgroup of vM. Further, let $K_0^w \subset \mathcal{O}_w$ be any field of the form (18). For example, if $wx = 0$, then we can take $K_0^w = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$. We will see later that if $w \neq v_{\mathcal{R}}$ then there always exists such a field K_0^w which properly contains $\mathcal{R}$.

Now we construct K_1^w as follows. Assume that $a \in K_0^w$ such that $\exp a \notin K_0^w$, but $v\exp a \in H_w$. Then by Lemma 3.5, $K_0^w(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is again of the form (18), with $vK_0^w(\exp a)^{\mathrm{rQ}\mathcal{F}} = vK_0^w \oplus \mathrm{Q}\, v\exp a \subset H_w$. The latter shows that it is again a subfield of $\mathcal{O}_w$. We repeat this procedure until we arrive at a field $K_1^w \subset \mathcal{O}_w$ of the form (18), which contains $\exp a$ for every $a \in K_0^w$ such that $\exp a \in \mathcal{O}_w^\times$. Then we construct K_2^w from K_1^w in the same way as we constructed K_1^w from K_0^w. We iterate to obtain fields $K_n^w \subset \mathcal{O}_w$, of the form (18). Their union

$$K_\infty^w := \bigcup_{n\in\mathbb{N}} K_n^w \subset \mathcal{O}_w$$

is $\mathrm{rQ}\mathcal{F}$-closed and of the form (18). By construction, we have:

Lemma 3.8 *K_∞^w is the uniquely determined smallest log- and $\mathrm{rQ}\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing K_0^w. It is of the form (18).*

We derive some further information from our construction.

Lemma 3.9 *Take $n \in \mathbb{N}$. If $a \in K_n^w$ with $va < 0$, $a > 0$, then*

$$v \log a \in vK_{n-1}^w \;, \quad \textit{and} \quad v \log_n a \in vK_0^w \;.$$

Proof: By the construction of K_n^w from K_{n-1}^w, there are elements $a_j \in K_{n-1}^w$, $j \in J$, such that $vK_n^w = vK_{n-1}^w \oplus \bigoplus_{j\in J} \mathbb{Q} v \exp a_j$. Hence, $a \in K_n^w$ can be written as

$$a = \prod_{j\in J_0} (\exp a_j)^{q_j} \cdot c \cdot r \cdot (1+\varepsilon)$$

with J_0 a finite subset of J, $q_j \in \mathbb{Q}$, $c \in K_{n-1}^w$, $r \in \mathbb{R}$ and $\varepsilon \in K_n^w$ with $v\varepsilon > 0$. Then $\log a = \sum_{j\in J_0} q_j a_j + \log c + \log r + \log(1+\varepsilon)$. Since $v \log a < 0$ by Lemma 2.4, but $v\log(1+\varepsilon) > 0$, we find that $v \log a = v(\sum_{j\in J_0} q_j a_j + \log c + \log r) \in vK_{n-1}^w$. By induction it follows that $v \log_n a \in vK_0^w$. □

If w is trivial on $\mathcal{R}$ and $wx = 0$ and we start our construction from $K_0^w = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$, then K_∞^w will be the uniquely determined smallest log- and rQ$\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing $\mathcal{R}(x)$. We denote it by

$$LE_{\mathcal{R},\mathcal{F}}^w(x) \;.$$

Let u denote the trivial valuation on M. Then $\mathcal{O}_u = M$ and $H_u = vM$. In this case, $LE_{\mathcal{R},\mathcal{F}}^u(x)$ is exp-closed and contains x. Therefore,

$$LE_{\mathcal{R},\mathcal{F}}^u(x) \;=\; LE_{\mathcal{R},\mathcal{F}}(x) \;.$$

Lemma 3.10 *Suppose that $x > \mathcal{R}$. Then for every $y \in LE_{\mathcal{R},\mathcal{F}}(x)$, $y > \mathcal{R}$, the sequence* $\exp_m y$, $m \geq 0$, *is cofinal in $LE_{\mathcal{R},\mathcal{F}}(x)$, and the sequence* $\log_m y$, $m \geq 0$, *is coinitial in* $\{z \in LE_{\mathcal{R},\mathcal{F}}(x) \mid z > \mathcal{R}\}$.

Proof: It suffices to show the result for $y = x$. Indeed, if it holds in this case, then there is $\nu \in \mathbb{N}$ such that $\exp_\nu x > y > \log_\nu x$. It follows that $\exp_n y > \exp_{\nu+n} x$, showing that also the sequence $\exp_m y$, $m \geq 0$, is cofinal. It also follows that $\log_n x > \log_{\nu+n} y$, showing that also the sequence $\log_m y$, $m \geq 0$, is coinitial.

Take any $a \in LE_{\mathcal{R},\mathcal{F}}(x)$, $x > \mathcal{R}$. From Lemma 3.9 with $w = u$ and $K_0^w = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ we infer that $v \log_n a \in v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ for some $n \in \mathbb{N}$. By Lemma 3.7, every element $\alpha < 0$ in this value group is either archimedean equivalent to vx, or satisfies $vx \ll \alpha < 0$. Since $v \log_n a \ll v \log_{n+1} a < 0$ by Lemma 2.4, it follows that $vx \ll v \log_{n+1} a < 0$. Hence by (1), $x > \log_{n+1} a$ and therefore, $\exp_{n+1} x > a$.

Now let $a \in LE_{\mathcal{R},\mathcal{F}}(x)$, $a > \mathcal{R}$. As before, $v \log_n a \in v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ for some $n \in \mathbb{N}$. As the sequence $v \log_m x$, $m \geq 0$, is cofinal in the negative part of this value group, there is some m_0 such that $v \log_n a < v \log_{m_0} x$. Hence by (1), $a \geq \log_n a > \log_{m_0} x$. □

Now we deduce our main theorem on the valuation theoretical structure of $LE_{\mathcal{R},\mathcal{F}}(x)$. If we take $\mathcal{F} = \mathcal{F}_T$ and $M = H(\mathcal{R})$, then $H(\mathcal{R}) = LE_{\mathcal{R},\mathcal{F}}(x)$ by what we have remarked in the introduction, and thus the theorem describes the structure of the Hardy field $H(\mathcal{R})$.

Theorem 3.11 *$LE_{\mathcal{R},\mathcal{F}}(x)$ is of the form*

$$\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} \text{ with } x_i > 0 \text{ and } v_{\mathcal{R}} x_i,\ i \in I,\ \mathbb{Q}\text{-linearly independent.} \tag{21}$$

Moreover,

$$LE_{\mathcal{R},\mathcal{F}}(x) v_{\mathcal{R}} \;=\; \mathcal{R}\ . \tag{22}$$

The elements x_i can be chosen so as to include x and $\log_m x$ for all $m \in \mathbb{N}$.

If $\mathcal{R} = \mathbb{R}$, then $LE_{\mathcal{R},\mathcal{F}}(x)$ has exponential rank 1, in the sense of [K–K2]. In general, $\operatorname{exprk} LE_{\mathcal{R},\mathcal{F}}(x) = \operatorname{exprk} \mathcal{R} + 1$.

Proof: By our construction, we get that $LE_{\mathcal{R},\mathcal{F}}(x)$ is of the form (18). Since $\mathcal{F} \subseteq \mathcal{F}_T$, we have that $LE_{\mathcal{R},\mathcal{F}}(x) \subseteq LE_{\mathcal{R},\mathcal{F}_T}(x)$. By definition of the valuation $v_{\mathcal{R}}$, its valuation ring is the convex hull of $\mathcal{R}$ in M. As $\mathcal{R}$ is an elementary submodel of $LE_{\mathcal{R},\mathcal{F}_T}(x)$, we can deduce from [D–L], p. 75, (1), that this valuation ring is $T(\exp)$-convex in $LE_{\mathcal{R},\mathcal{F}_T}(x)$. Since the $T(\exp)$-definable closure of $\mathcal{R}(x)$ in $LE_{\mathcal{R},\mathcal{F}_T}(x)$ is equal to $LE_{\mathcal{R},\mathcal{F}_T}(x)$, we can apply Corollary 5.4 of [D–L] to obtain that $LE_{\mathcal{R},\mathcal{F}_T}(x) v_{\mathcal{R}} = \mathcal{R}$. Since $\mathcal{R} \subset LE_{\mathcal{R},\mathcal{F}}(x) \subseteq LE_{\mathcal{R},\mathcal{F}_T}(x)$, this proves (22). By Lemma 3.6, this also implies that $v_{\mathcal{R}} x_i$, $i \in I$, are $\mathbb{Q}$-linearly independent.

The exponential rank is the order type of the set of proper $T(\exp)$-convex valuation rings, ordered by inclusion. Lemma 3.10 shows that $LE_{\mathcal{R},\mathcal{F}}(x)$ has exactly one more than $\mathcal{R}$, namely $\mathcal{R}$ itself. This proves our assertions about the exponential rank. □

3.4 Levels

An infinitely increasing unary function f on $\mathcal{R}$ **has level** s if $s \in \mathbb{Z}$ and there is $N \in \mathbb{N}$ such that $\log_{N+s} \circ f$ is asymptotic to $\log_N$ on $\mathcal{R}$. Note that if the latter holds, then it also holds for every integer $N' > N$ in the place of N. If a denotes the germ of f in $H(\mathcal{R})$, then by (15) the condition is equivalent to

$$v\left(\frac{\log_{N+s} a}{\log_N x} - 1\right) \;>\; v\mathcal{R}\ .$$

Here, N can be chosen such that $N + s \geq 0$. Suppose that $s < s' \in \mathbb{Z}$. Since $a > \mathcal{R}$ we have that $va < v\mathcal{R}$; hence by Corollary 2.5, $v \log_{N+s} a \neq v \log_{N+s'} a$ which shows that the above inequality cannot hold for s' in the place of s. Thus, the level s is uniquely determined (see also [M–M]).

We say that $\mathcal{R}$ is **levelled** if every $\mathcal{R}$-definable ultimately strictly increasing and unbounded unary function on $\mathcal{R}$ has a level. In this section, we will prove that every definable function on $\mathcal{R}$ has a level, and we will determine this level explicitly.

Take any $a \in LE_{\mathcal{R},\mathcal{F}}(x)$ such that $a > \mathcal{R}$. According to our construction, we write $LE_{\mathcal{R},\mathcal{F}}(x) = K_\infty$ with $K_0 = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$. By Lemma 3.9 there is some $n \in \mathbb{N}$ such that $v \log_n a \in vK_0$. Similarly as in the proof of Lemma 3.4, we write

$\log_n a = r' \prod_{i\geq 0}(\log_i x)^{q_i} \cdot r \cdot (1+\varepsilon)$ with $q_i \in \mathbb{Q}$, only finitely many of them nonzero, $r' \in \mathcal{R}$, $r \in \mathbb{R}$ and $\varepsilon \in K$ such that $v\varepsilon > 0$. It follows that

$$\log_{n+1} a \;=\; \log r' + \sum_{i\geq 0} q_i \log_{i+1} x \;+\; \log r \;+\; \log(1+\varepsilon)\;.$$

As $a > \mathcal{R}$ by assumption, there must be at least one nonzero q_i. Let i_0 be the smallest of all $i \geq 0$ for which $q_i \neq 0$. We have that $v \log r = 0$, $v\log(1+\varepsilon) > 0$ and $v \log_{i_0+1} x < v \log_{i+1} x$ for $i > i_0$. Also, $v \log_{i_0+1} x < vr'$. Thus, we can write $\log_{n+1} a = q_{i_0} \log_{i_0+1} x \cdot (1+\varepsilon')$ with $v\varepsilon' > 0$. Then

$$\log_{n+2} a \;=\; \log q_{i_0} \;+\; \log_{i_0+2} x \;+\; \log(1+\varepsilon')\;.$$

Again, $v \log_{i_0+2} x < 0 = v \log q_{i_0} < v\varepsilon' = v\log(1+\varepsilon')$. Hence,

$$v\left(\log_{n+2} a \;-\; \log_{i_0+2} x\right) \;=\; v\left(\log q_{i_0} + \log(1+\varepsilon')\right) \;=\; v \log q_{i_0} \;=\; 0\;.$$

Thus,

$$v\left(\frac{\log_{n+2} a}{\log_{i_0+2} x} \;-\; 1\right) \;=\; -v \log_{i_0+2} x \;>\; v\mathcal{R}\;. \tag{23}$$

We have now proved a result which in fact constitutes an abstract notion of levels, without referring to Hardy fields:

Proposition 3.12 *Take any element $a \in LE_{\mathcal{R},\mathcal{F}}(x)$ such that $a > \mathcal{R}$. Then a "has level over $\mathcal{R}$" in the following sense: there is some $s \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that*

$$v_{\mathcal{R}}(\log_{N+s} a \;-\; \log_N x) \;>\; v_{\mathcal{R}} \log_N x\;.$$

Now take any $\mathcal{R}$-definable, ultimately strictly increasing and unbounded function f on $\mathcal{R}$. Let a be the germ of f at infinity. Then $a > \mathcal{R}$. Hence, a is an element of the Hardy field $H(\mathcal{R}) = LE_{\mathcal{R},\mathcal{F}_T}(x)$ of $\mathcal{R}$ (where $x > \mathcal{R}$). Then (23) shows that $\log_{n+2} f(x)$ and $\log_{i_0+2} x$ are asymptotic as functions on $\mathcal{R}$. That is,

the function f has level $n - i_0$.

This proves Theorem 1.1.

3.5 A maximality property of the T-definable closure in the $T(\exp)$-definable closure

Lemma 3.13 *Assume that T has field of exponents $\mathbb{R}$ and that $\mathbb{R} \subset \mathcal{R} \subset M$ are models of $T(\exp)$. Let $x \in M$, $x > \mathcal{R}$. Then $\mathcal{R}(x)^{\mathcal{F}_T}$ (the T-definable closure of $\mathcal{R} \cup \{x\}$ in M) has the following maximality property:*
1) $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \simeq \mathbb{R}$,
2) $\mathcal{R}(x)^{\mathcal{F}_T}$ is maximal among all subfields of $LE_{\mathcal{R},\mathcal{F}_T}(x)$ whose value group w.r.t. $v_{\mathcal{R}}$ is archimedean.

Proof: Assertion 1) follows from Lemma 3.2. In order to prove assertion 2), we show the following: Take any $a \in LE_{\mathcal{R},\mathcal{F}_T}(x) \setminus \mathcal{R}(x)^{\mathcal{F}_T}$. Then $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}(a)$ is not archimedean.

By Theorem 3.11 we can write $LE_{\mathcal{R},\mathcal{F}_T}(x) = \mathcal{R}(x_i \mid i \in I)^{\mathcal{F}_T}$ with $x_i > 0$ and $v_{\mathcal{R}}x_i$, $i \in I$, $\mathbb{R}$-linearly independent, and x among the x_i. As $a \in \mathcal{R}(x_i \mid i \in I)^{\mathcal{F}_T}$, there are $x_{i_1}, \ldots, x_{i_n}$ ($n \geq 1$) such that $a \in \mathcal{R}(x, x_{i_1}, \ldots, x_{i_n})^{\mathcal{F}_T}$, and we choose n minimal with this property. By the Exchange Lemma for o-minimal theories ([P–S]) applied to T, we then obtain that

$$x_{i_1} \in \mathcal{R}(x, a, x_{i_2}, \ldots, x_{i_n})^{\mathcal{F}_T} \ . \tag{24}$$

Suppose that $v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T} = v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}$. Then by Lemma 3.2,

$$\begin{aligned} v_{\mathcal{R}}\mathcal{R}(x, a, x_{i_2}, \ldots, x_{i_n})^{\mathcal{F}_T} &= v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T}(x_{i_2}, \ldots, x_{i_n})^{\mathcal{F}_T} \;=\; v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T} \oplus \bigoplus_{j=2}^{n} \mathbb{R}v_{\mathcal{R}}x_{i_j} \\ &= v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \oplus \bigoplus_{j=2}^{n} \mathbb{R}v_{\mathcal{R}}x_{i_j} \;=\; \mathbb{R}v_{\mathcal{R}}x \oplus \bigoplus_{j=2}^{n} \mathbb{R}v_{\mathcal{R}}x_{i_j} \ . \end{aligned}$$

But this does not contain $v_{\mathcal{R}}x_{i_1}$. This contradiction to (24) shows that

$$v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T} \;\neq\; v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \ .$$

By the Valuation Property ([D–S2], Proposition 9.2) it follows that

$$v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \;\subsetneq\; v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}(a) \ .$$

Since $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \simeq \mathbb{R}$ it follows that $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}(a)$ is not archimedean. □

Lemma 3.14 *Let $H \subset H(\mathcal{R})$ be a subfield containing $\mathcal{R}(x)$ and closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs (i.e., germs $a \in H$ such that $a > \mathcal{R}$). If H is polynomially bounded (i.e., every germ in H is bounded by a power x^n for some $n \in \mathbb{N}$), then $v_{\mathcal{R}}(H)$ is archimedean.*

Proof: Assume for a contradiction that there is $g \in H(\mathcal{R})$ such that $g > \mathcal{R}$ and $v_{\mathcal{R}}g \ll v_{\mathcal{R}}x$ or $v_{\mathcal{R}}x \ll v_{\mathcal{R}}g$. The former implies that $g > x^n$ for all $n \in \mathbb{N}$, a contradiction to the fact that H is polynomially bounded. So assume that $v_{\mathcal{R}}x \ll v_{\mathcal{R}}g$. But this implies that for all $n \in \mathbb{N}$,

$$x^n < g^{-1} \ ,$$

where g^{-1} denotes the compositional inverse of g. This again contradicts the assumption that H is polynomially bounded. Indeed, let $n \in \mathbb{N}$. Since $g^n < x$, there exists $r \in \mathcal{R}$ (and we may assume $r > 1$) such that for $a \in \mathcal{R}$ with $a > r$ we have $g(a)^n < a$. On the other hand, g is invertible, ultimately. So for b large enough, $g^{-1}(b) = a$ exists with $a > r$. Thus, $g(g^{-1}(b))^n < g^{-1}(b)$. □

Corollary 3.15 *The field $\mathcal{R}(x)^{\mathcal{F}_T}$ (i.e., the Hardy field associated with the reduct of $\mathcal{R}$ to the language of T) is maximal among the polynomially bounded subfields of $H(\mathcal{R})$ which are closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs.*

Proof: Let H be a polynomially bounded subfield of $H(\mathcal{R})$ closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs, and containing $\mathcal{R}(x)^{\mathcal{F}_T}$. Then by Lemma 3.14, $v_{\mathcal{R}}H$ is archimedean. Hence by Lemma 3.13, H cannot be a proper extension of $\mathcal{R}(x)^{\mathcal{F}_T}$. □

Let us note that there exist polynomially bounded subfields of $H(\mathcal{R})$ which properly contain $\mathcal{R}(x)^{\mathcal{F}_T}$. For instance, $\mathcal{R}(x, \log x)^{\mathcal{F}_T}$ and $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathcal{F}_T}$ are such fields. But they are not closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs.

3.6 A maximality property of the Hardy field $H(\mathcal{R}_{\text{an,powers}})$

Now we consider the special case where $\mathcal{F}_T$ is the set of function symbols for 0-definable functions in $\mathbb{R}_{\text{an,powers}}$. We let $\mathcal{R}_{\text{an,powers}}$ denote the reduct of $\mathcal{R}$ to the language of $\mathbb{R}_{\text{an,powers}}$, and $\mathcal{R}_{\text{an,exp}}$ the reduct of $\mathcal{R}$ to the language of $\mathbb{R}_{\text{an,exp}}$. Since

$$x^r \;=\; \exp(r \log x)$$

for all $r \in \mathbb{R}$, the power functions are $\mathbb{R}$-definable (actually, already 0-definable) in $\mathcal{R}_{\text{an,exp}}$. Therefore,

$$H(\mathcal{R}_{\text{an,exp}}) \;=\; H(\mathcal{R}) \;.$$

On the other hand, $H(\mathcal{R}_{\text{an,powers}})$ is a proper subfield of $H(\mathcal{R})$. It has the following maximality property:

Theorem 3.16 *Let $H \subseteq H(\mathcal{R})$ be a polynomially bounded field containing $H(\mathcal{R}_{\text{an,powers}})$ and closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs. Then $H = H(\mathcal{R}_{\text{an,powers}})$.*

In particular, $H(\mathcal{R}_{\text{an,powers}})$ is maximal among the Hardy subfields of $H(\mathcal{R})$ associated with polynomially bounded reducts of $\mathcal{R}$.

Proof: We take T to be the elementary theory of $\mathcal{R}_{\text{an,powers}}$. We know that $H(\mathcal{R}_{\text{an,powers}}) = \mathcal{R}(x)^{\mathcal{F}_T}$ with $x \in H(\mathcal{R})$, $x > \mathcal{R}$ the germ of the identity function. Now our first assertion follows from Corollary 3.15.

If H is the Hardy field of a polynomially bounded reducts of $\mathcal{R}$, then H is closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs. Hence our second assertion follows from the first. □

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