Towers of complements to valuation rings and truncation closed embeddings of valued fields *

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Abstract

We study necessary and sufficient conditions for a valued field $\mathbb{K}$ with value group G and residue field $\mathbf{k}$ (with char $\mathbb{K}$ = char $\mathbf{k}$) to admit a truncation closed embedding in the field of generalized power series $\mathbf{k}((G, f))$ (with factor set f). We show that this is equivalent to the existence of a family (*tower of complements*) of $\mathbf{k}$-subspaces of $\mathbb{K}$ which are complements of the (possibly fractional) ideals of the valuation ring, and satisfying certain natural conditions. If $\mathbb{K}$ is a Henselian field of characteristic 0 or, more generally, an algebraically maximal Kaplansky field, we give an intrinsic construction of such a family which does not rely on a given truncation closed embedding. We also show that towers of complements and truncation closed embeddings can be extended from an arbitrary field to at least one of its maximal immediate extensions.

1 Introduction

Truncation closed embeddings of valued fields in fields of generalized power series (see Section 2.2 for definitions and notations) were introduced by Mourgues and Ressayre in their investigation of integer parts of ordered fields. An

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integer part (IP for short) Z of an ordered field $\mathbb{K}$ is a discretely ordered subring, with 1 as the least positive element, and such that for every $x \in \mathbb{K}$, there is a $z \in Z$ such that $z \leq x < z+1$. The interest in studying these rings originates from Shepherdson's work in [S], who showed that IP's of real closed fields are precisely the models of a fragment of Peano Arithmetic called Open Induction. In [M-R], the authors establish the existence of an IP for any real closed field $\mathbb{K}$ as follows (see Section 2.2 for definitions and notations): Let v be the natural valuation on $\mathbb{K}$. Denote by $\mathbf{k}$ the residue field and by G the value group of $\mathbb{K}$. Fix a residue field section ι (we will assume that ι is the identity), and a value group section $t : G \to \mathbb{K}^\star$. [M–R] show that there is an order preserving embedding φ of $\mathbb{K}$ in the field of generalized power series $\mathbf{k}((G))$ such that $\varphi(\mathbb{K})$ is a truncation closed subfield. They observe that for the field $\mathbf{k}((G))$, an integer part is given by $\mathbf{k}((G^{<0})) \oplus \mathbb{Z}$, where $\mathbf{k}((G^{<0}))$ is the (non-unital) $\mathbf{k}$-algebra of power series with negative support. It follows that for any truncation closed subfield F of $\mathbf{k}((G))$, an integer part is given by $Z_F = (\mathbf{k}((G^{<0})) \cap F) \oplus \mathbb{Z}$. Finally $\varphi^{-1}(Z_F)$ is an integer part of $\mathbb{K}$ if we take $F = \varphi(\mathbb{K})$. Let A be a $\mathbf{k}$-subspace of $\mathbb{K}$ which is also an additive complement to the valuation ring $\mathcal{O}$ of $\mathbb{K}$: we will call A a **k-complement of** $\mathcal{O}$. We say that A is **multiplicative** if $A \cdot A \subseteq A$ (i.e. if A is a $\mathbf{k}$-algebra). A multiplicative $\mathbf{k}$-complement of $\mathcal{O}$ will be called a **k-algebra complement of** $\mathcal{O}$. Since $\mathbf{k}((G^{<0})) \cap F$ is clearly a multiplicative $\mathbf{k}$-complement of the valuation ring of F, $\varphi^{-1}(\mathbf{k}((G^{<0})) \cap F)$ is a multiplicative $\mathbf{k}$-complement of $\mathcal{O}$. We call an integer part Z of $\mathbb{K}$ (respectively a multiplicative $\mathbf{k}$-complement of $\mathcal{O}$) obtained in this way from a truncation closed embedding a **truncation integer part** of $\mathbb{K}$ (respectively a **truncation k-algebra complement of** $\mathcal{O}$, or **truncation k-algebra** for short). In this terminology, it follows in particular from [M-R] that every real closed field $\mathbb{K}$ with residue field $\mathbf{k}$ and valuation ring $\mathcal{O}$ admits a truncation IP and a truncation $\mathbf{k}$-algebra complement of $\mathcal{O}$.

In light of these results, we asked in Remarks 3.2 and 3.3 of [B-K-K] more generally for necessary and sufficient conditions on the valued field $\mathbb{K}$ for the existence of $\mathbf{k}$-algebra complements of $\mathcal{O}$. We also asked whether such complements are always truncation $\mathbf{k}$-algebra complements. A valued field $\mathbb{K}$ with residue field $\mathbf{k}$ and value group G is called a **Kaplansky field** if $\mathbf{k}$ and G satisfy a pair of conditions called Kaplansky's "Hypothesis A" ([Kap]; page 312 statements (1) and (2)). The conditions are void if the characteristic of $\mathbf{k}$ is 0, whereas if the characteristic of $\mathbf{k}$ is $p > 0$, then Hypothesis A holds if and only if G is p-divisible and $\mathbf{k}$ does not admit any extensions of degree divisible by p. Kaplansky's original result [Kap] Theorem 6[1] yields that a Kaplansky field $\mathbb{K}$ (with char $\mathbb{K}$ = char $\mathbf{k}$) can be embedded in the power series field $\mathbf{k}((G, f))$, possibly with a suitable choice of a factor set f (cf. Definition 2.14). In view of this, it is natural to ask more generally for necessary and sufficient conditions on a valued field $\mathbb{K}$ for the existence of truncation closed embeddings of $\mathbb{K}$ in $\mathbf{k}((G, f))$. This paper addresses these questions.

Unless otherwise stated, we assume throughout that the valued field $\mathbb{K}$ admits a value group section and a residue field section. In particular, we only deal with the "equal characterisitic" case, i.e., char $\mathbb{K}$ = char $\mathbf{k}$.

The main tool in our investigations is the following concept. A tower of complements of the valuation ring $\mathcal{O}$ of $\mathbb{K}$ is a family $\mathcal{A} = \left\{ A[\Lambda] : \Lambda \in \breve{G} \right\}$ of $\mathbf{k}$-subspaces of $\mathbb{K}$ (indexed by the order completion $\breve{G}$ of the value group G; cf. Definition 2.1) satisfying certain natural properties (cf. Definition 3.2). We show (cf. Theorem 3.15) that the existence of such a family is a necessary and sufficient condition for a valued field $(\mathbb{K}, v)$ to admit a truncation closed embedding in the field of generalized power series $\mathbf{k}((G, f))$, and that there is a one-to-one correspondence between towers of complements for $\mathbb{K}$ and truncation closed embeddings of $\mathbb{K}$ in $\mathbf{k}((G, f))$ (cf. Corollary 3.16.) A tower of complements $\mathcal{A}$ is uniquely determined by $\mathbf{A} := A[0^-] \in \mathcal{A}$ which is a $\mathbf{k}$-algebra complement of $\mathcal{O}$ (cf. Corollary 3.5). Conversely, given $\mathbf{A}$ any $\mathbf{k}$-algebra complement of $\mathcal{O}$, we conclude that $\mathbf{A}$ is a truncation $\mathbf{k}$-algebra complement if and only if there is a tower of complements $\mathcal{A}$ such that $\mathbf{A} = A[0^-]$. In Section 4 we analyze when such an $\mathcal{A}$ can be constructed for a given $\mathbf{A}$.

In Section 5, we analyze the procedure for extending towers of complements (T.o.C.). We start with the subfield of rational series $\mathbf{k}(G) \subseteq \mathbb{K}$ and proceed by induction, building T.o.C.'s for larger and larger subfields of $\mathbb{K}$. The field of rational series $\mathbf{k}(G)$ has a T.o.C. (Section 5.1). We proceed by induction: we assume that we already built a T.o.C. $\mathcal{A}$ for a subfield K of $\mathbb{K}$, such that $\mathbf{k}(G) \subseteq K$. Note that $\mathbb{K}$ is an immediate extension of K. Let $a \in \mathbb{K} \setminus K$. We want to extend the T.o.C. $\mathcal{A}$ to a T.o.C. $\mathcal{B}$ for $K(a)$. Let $(a_\nu)_{\nu \in I}$ be a pseudo Cauchy sequence in K with limit a. We assume that a satisfies one

[1] There is a misprint on line 2 of the statement of the Theorem; K should be replaced by $\mathfrak{K}$.

of the two conditions a) or b) of the Fundamental Hypothesis 5.9. In case b), the algebraic case, we can extend $\mathcal{A}$ to a T.o.C. $\mathcal{B}$ for $K(a) = K[a]$ if certain conditions are satisfied; the extension $\mathcal{B}$ is defined in Definition 5.11. In particular, we can extend $\mathcal{A}$ to the Henselization of K (Lemma 5.24). In case a), the transcendental case, we can extend $\mathcal{A}$ to a T.o.C. $\mathcal{B}$ for $K[a]$ (Corollary 5.21); note that in this case $K[a]$ is a ring, not a field. Then we extend $\mathcal{B}$ further to a T.o.C. for $K(a)$, the quotient field of $K[a]$.

Putting these steps together, we prove that every Henselian field of residue characteristic 0 has a T.o.C. . In fact, start with $\mathbf{k}(\mathrm{G})$, pass to the Henselization, add a transcendental element a satisfying a), pass again to the Henselization, and so on. Here we use that Henselian fields of residue characteristic 0 do not admit proper immediate algebraic extensions. But in the case of positive residue characteristic, one has to deal with such extensions. In Section 5.4 we prove that towers of complements on a given field of positive characteristic can be extended to *at least one* maximal immediate algebraic extension and thus to *at least one* maximal immediate extension (Theorem 5.28). It follows from our results of Sections 5.3 and 5.4 that, in the equal characterisitc case, algebraically maximal Kaplansky fields admit towers of complements and thus also truncation closed embeddings in power series fields (cf. Theorem 5.25, Theorem 5.29 and Corollary 5.30). Without the condition "algebraically maximal", the existence of truncation closed embeddings can in general not be expected (cf. examples in [F] and [Ku2]).

Note that Fornasiero ([F]; Theorem 5.1) already showed the existence of truncation closed embeddings for Henselian fields of residue characteristic 0, and has indicated the same result (in the equal positive characteristic case) for algebraically maximal Kaplansky fields ([F]; paragraph following Theorem 8.12), by generalizing the approach of [M-R]. But in this paper we prove it through an intrinsic construction of towers of complements. This approach allows us to obtain an even stronger result. Namely, Theorem 5.28 implies that a truncation closed embedding of a field of positive characteristic can be extended to a truncation closed embedding of at least one of its maximal immediate algebraic extensions and at least one of its maximal immediate extensions, even if the field is not a Kaplansky field. In that case, the truncation closed embedding may not be extendable to all such extensions, as we show in an example. This means that for such fields, there are maximal immediate extensions that are "better" than others. It should definitely be interesting to study their properties, both from an algebraic and from a model theoretic point of view.

We conclude with the following remark and open question: There exist valued fields (with sections for the value group G and the residue field $\mathbf{k}$) that admit a valuation preserving embedding (compatible with the sections) in $\mathbf{k}((G, f))$, but admit *no* truncation closed embedding (cf. examples in [F] and [Ku2]). We do not know whether such fields can be algebraically maximal.

2 Preliminaries

2.1 Dedekind cuts on ordered groups

Let O be an ordered set. A **cut** $\left(\Lambda^L, \Lambda^R\right)$ of O is a partition of O into two subsets Λ^L and Λ^R, such that, for every $\lambda \in \Lambda^L$ and $\lambda' \in \Lambda^R$, $\lambda < \lambda'$. Λ^L is an **initial segment** of O, and Λ^R is a **final segment** of O. We will denote with $\breve{O}$ the set of cuts $\Lambda := \left(\Lambda^L, \Lambda^R\right)$ of the order O (including $-\infty := \left(\emptyset, O\right)$ and $+\infty := \left(O, \emptyset\right)$).

Unless specified otherwise, small Greek letters $\gamma, \lambda, \cdots$ will range among elements of O, capital Greek letters $\Gamma, \Lambda, \cdots$ will range among elements of $\breve{O}$.

Given $\gamma \in O$,

$$\gamma^- := \big((-\infty, \gamma),\, [\gamma, +\infty)\big) \quad \text{and}$$
$$\gamma^+ := \big((-\infty, \gamma],\, (\gamma, +\infty)\big)$$

are the cuts determined by it. Note that Λ^R has a minimum λ if and only if $\Lambda = \lambda^-$. Dually, Λ^L has a maximum γ if and only if $\Lambda = \gamma^+$.

The **ordering** on $\breve{O}$ is given by $\Lambda \leq \Gamma$ if and only if $\Lambda^L \subseteq \Gamma^L$ (or, equivalently, $\Lambda^R \supseteq \Gamma^R$). To simplify the notation, we will sometimes write $\gamma < \Lambda$ as a synonym of $\gamma \in \Lambda^L$, or equivalently $\gamma^- < \Lambda$, or equivalently $\gamma^+ \leq \Lambda$. Similarly, $\gamma > \Lambda$ if and only if $\gamma \in \Lambda^R$, or equivalently $\gamma^+ > \Lambda$. Hence, we have $\gamma^- < \gamma < \gamma^+$.

An ordered set O is **complete** if for every $S \subseteq O$, the l.u.b. and the g.l.b. of S exist. Note that if O is any ordered set, then $\breve{O}$ is complete. Given a subset $S \subseteq O$, $S^+ \in \breve{O}$ is the smallest cut Λ such that $S \subseteq \Lambda^L$, and S^- is the largest cut Γ such that $S \subseteq \Gamma^R$. Note that $S^+ = -\infty$ if and only if S is empty, and $S^+ = +\infty$ if and only if S is unbounded. Note also that $S^+ = \sup\left\{ \gamma^+ : \gamma \in S \right\}$,[2] and $S^+ > \gamma$ for every $\gamma \in S$. Moreover, $\Lambda = (\Lambda^L)^+$.

[2]The supremum on the R.H.S. is taken in the ordered set $\breve{O}$.

Now let G be an ordered Abelian group. Given $\Lambda, \Gamma \in \breve{G}$, their *left* **sum** is the cut

$$\Lambda + \Gamma := \left\{ \lambda + \gamma : \lambda < \Lambda, \gamma < \Gamma \right\}^{+}.$$

We also define *right* **sum**, as

$$\Lambda \mathbin{+^{\mathrm{R}}} \Gamma := \left\{ \lambda + \gamma : \lambda > \Lambda, \gamma > \Gamma \right\}^{-}.$$

Given $\gamma \in G$, we write

$$\gamma + \Lambda := \left(\left\{ \gamma + \lambda : \lambda \in \Lambda^{L} \right\}, \left\{ \gamma + \lambda' : \lambda' \in \Lambda^{R} \right\} \right).$$

One can verify that $\gamma + \Lambda = \gamma^{+} + \Lambda = \gamma^{-} \mathbin{+^{\mathrm{R}}} \Lambda$, and that:

$$\begin{aligned}
\Lambda &+ (+\infty) &= \quad \Lambda &\mathbin{+^{\mathrm{R}}} (+\infty) &= +\infty,\\
\Lambda &+ (-\infty) &= \quad \Lambda &\mathbin{+^{\mathrm{R}}} (-\infty) &= -\infty,\\
+\infty &+ (+\infty) &= +\infty &\mathbin{+^{\mathrm{R}}} (+\infty) &= +\infty,\\
-\infty &+ (-\infty) &= -\infty &\mathbin{+^{\mathrm{R}}} (-\infty) &= -\infty.
\end{aligned}$$

Note also that $\Lambda + \Gamma \leq \Lambda \mathbin{+^{\mathrm{R}}} \Gamma$.

2.1 Remark. $(\breve{G}, \leq)$ is a complete linear order, and moreover $(\breve{G}, +, \leq)$ is an ordered commutative monoid, with neutral element 0^{+}, that is, if $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$. Similarly $(\breve{G}, \mathbin{+^{\mathrm{R}}}, \leq)$ is an ordered commutative monoid, with neutral element 0^{-}. The map ϕ^{+} (resp. ϕ^{-}) from $(G, \leq, 0, +)$ to $(\breve{G}, \leq, 0^{+}, +)$ (resp. to $(\breve{G}, \leq, 0^{-}, \mathbin{+^{\mathrm{R}}})$) sending γ to γ^{+} (resp. to γ^{-}) is a homomorphism of ordered monoids. The anti-isomorphism $-$ of $(G, \leq)$, sending γ to $-\gamma$, induces an anti-isomorphism (with the same name $-$) between $(\breve{G}, \leq, +)$ and $(\breve{G}, \geq, \mathbin{+^{\mathrm{R}}})$, sending Λ to $\left(-\Lambda^{R}, -\Lambda^{L}\right)$. Hence, all theorems about $+$ have a dual statement about $\mathbin{+^{\mathrm{R}}}$.

2.2 Remark. Note that $-(\gamma^{+}) = (-\gamma)^{-}$, and $-(\gamma^{-}) = (-\gamma)^{+}$.

2.3 Definition. Given $\Lambda, \Gamma \in \breve{G}$, define their (right) difference $\Lambda - \Gamma$ in the following way:

$$\Lambda - \Gamma := \left\{ \lambda - \gamma : \lambda > \Lambda, \gamma < \Gamma \right\}^{-}.$$

The following Lemma is easily proved (see Lemma 2.11 and Remark 3.6 of [F-M]).

2.4 Lemma. *1.* $\Lambda < \Theta$ *if and only if* $\Lambda - \Theta < 0$.

2. $\Lambda \geq \Theta$ *if and only if* $\Lambda - \Theta > 0$.

3. $\Lambda > -\Theta$ *if and only if* $\Lambda + \Theta > 0$.

4. $\Lambda < \Gamma + \Theta$ *if and only if* $\Lambda - \Gamma < \Theta$.

5. $\Lambda \geq \Gamma + \Theta$ *if and only if* $\Lambda - \Gamma \geq \Theta$.

6. $\Lambda \geq (-\Gamma) + \Theta$ *if and only if* $\Lambda \overset{\mathrm{R}}{+} \Gamma \geq \Theta$.

2.5 Definition. Given $n \in \mathbb{N}$, define

$$\Lambda - n\Gamma := \Lambda \underbrace{-\Gamma - \cdots - \Gamma}_{n \text{ times}}, \text{ and } \Lambda + n\Gamma := \Lambda \underbrace{+\Gamma + \cdots \Gamma}_{n \text{ times}}.$$

In particular, $\Lambda - 0\Gamma = \Lambda + 0\Gamma = \Lambda$. Moreover, given $n \in \mathbb{N}^\star$, define

$$n\Gamma := \underbrace{\Gamma + \cdots \Gamma}_{n \text{ times}}, \text{ and } (-n)\Gamma := -(n\Gamma) = \underbrace{-\Gamma - \cdots - \Gamma}_{n \text{ times}}.$$

The following technical results will be used throughout the paper, and are given without proof (see Proposition 3.15 and Corollary 3.16 of [F-M]).

2.6 Proposition. $(\Lambda - n\Gamma) + n\Gamma \leq \Lambda \leq (\Lambda + n\Gamma) - n\Gamma$.

2.7 Corollary. $(\Lambda - n\Gamma) + (\Lambda' - n'\Gamma) \leq (\Lambda + \Lambda') - (n + n')\Gamma$.

2.8 Corollary. *Let* $d, k, m \in \mathbb{N}$, *with* $k < m$. *Then,*

$$\big(\Lambda - (m + d)\Gamma\big) + (m\Gamma - k\Gamma) \leq \Lambda - (d + k)\Gamma.$$

2.9 Lemma. *For every* $i, j, k, m, d \in \mathbb{N}^\star$ *such that* $i, j, k < m$, *and* $i + j = m + d$,

$$(\Lambda - i\Gamma) + (\Lambda' - j\Gamma) + (m\Gamma - k\Gamma) \leq (\Lambda + \Lambda') - (d + k)\Gamma.$$

Proof. The lemma is a consequence of corollaries 2.8 and 2.7. In fact,

$$\begin{aligned}(\Lambda - i\Gamma) + (\Lambda' - j\Gamma) + (m\Gamma - k\Gamma) \leq \\ \leq \big((\Lambda + \Lambda') - (m + d)\Gamma\big) + (m\Gamma - k\Gamma) \leq (\Lambda + \Lambda') - (d + k)\Gamma. \quad \square\end{aligned}$$

Set $\widehat{\Gamma} := \Gamma - \Gamma$. It is straightforward to verify that $\hat{\Gamma} - \hat{\Gamma} = \hat{\Gamma} = \hat{\Gamma} + \hat{\Gamma}$, and to establish the following;

2.10 Remark. If Γ is of the form $\gamma + \widehat{\Gamma}$, then $m\Gamma - k\Gamma = (m - k)\Gamma$ for every $k < m \in \mathbb{N}$.

2.2 Valued Fields

We need to recall some facts about valued fields. (Cf. [E-P], [E], [Ri]).

Let $\mathbb{K}$ be a field, G an ordered Abelian group and ∞ an element greater than every element of G. A surjective map $v : \mathbb{K} \to G \cup \{\infty\}$ is a **valuation** on $\mathbb{K}$ if for all $a, b \in \mathbb{K}$: (i) $v(a) = \infty$ if and only if $a = 0$, (ii) $v(ab) = v(a) + v(b)$, (iii) $v(a - b) \geq \min\{v(a), v(b)\}$. We say that $(\mathbb{K}, v)$ is a **valued field**, and shall write just $\mathbb{K}$ whenever the context is clear. It follows that $v(a - b) = \min\{v(a), v(b)\}$ if $v(a) \neq v(b)$. The **value group** of $\mathbb{K}$ is $v(\mathbb{K}) := G$. The **valuation ring** of v is $\mathcal{O} := \{a \,;\, a \in \mathbb{K} \text{ and } v(a) \geq 0\}$ and the **valuation ideal** is $\mathcal{M} := \{a \,;\, a \in \mathbb{K} \text{ and } v(a) > 0\}$. The field $\mathcal{O}/\mathcal{M}$, denoted by $\mathbf{k}$, is the **residue field**. For $b \in \mathcal{O}$, $\bar{b}$ is its image under the residue map.

A valued field $\mathbb{K}$ is **Henselian** if it satisfies Hensel's Lemma: given a polynomial $p(x) \in \mathcal{O}[x]$, and $a \in \mathbf{k}$ a simple root of the reduced polynomial $\overline{p(x)} \in \mathbf{k}[x]$, we can find a root $b \in \mathbb{K}$ of $p(x)$ such that $\bar{b} = a$.

Let $\mathbb{K}$ be a valued field, with value group G and residue field $\mathbf{k}$, with the same characteristic as $\mathbb{K}$. Let $\mathcal{O}$ be the valuation ring of $\mathbb{K}$, and $\mathcal{M}$ its maximal ideal. A **value group section** is a map $t : G \to \mathbb{K}^\star$ such that $\forall \gamma \in G \quad v(t^\gamma) = \gamma$ and $t^{-\gamma} = 1/t^\gamma$. Note that t must satisfy $t^0 = 1$. Additional conditions on t might be imposed later. A **residue field section** is a field embedding $\iota : \mathbf{k} \to \mathbb{K}$ such that $\overline{\iota x} = x$ for every $x \in \mathbf{k}$. Whenever the context is clear, we will just write section to refer to either a value group section or a residue field section. Further, we will assume that ι is the identity. We recall the definition of a generalized power series fields with factor set.

2.11 Definition. Let $(A, +, 0)$ and $(B, \cdot, 1)$ be two Abelian groups. Then a 2 **co-cycle** is a map $f : A \times A \to B$ satisfying the following conditions:

1. $f[\alpha, \beta] = f[\beta, \alpha]$.
2. $f[0, 0] = f[0, \alpha] = f[\alpha, 0] = 1$.
3. $f[\alpha, \beta + \gamma] f[\beta, \gamma] = f[\alpha + \beta, \gamma] f[\alpha, \beta]$.
4. $f[-\alpha, \alpha] = 1$.

The following is easily verified:

2.12 Lemma. *Given a value group section t, the map $f : G \times G \to \mathbb{K}^\star$ defined by*

$$dt[\alpha, \beta] := f[\alpha, \beta] := \frac{t^\alpha\, t^\beta}{t^{(\alpha+\beta)}}$$

is a 2 co-cycle. Moreover, t is a group homomorphism if and only if $f = 1$.

The co-cycle obtained from the section t in Lemma 2.12 is denoted by $f := dt$.

2.13 Definition (Factor set)**.** Let $\mathbb{K}$ be a valued field containing its residue field $\mathbf{k}$. A **factor set** is a 2 co-cycle $f : G \times G \to \mathbf{k}^\star$.

If $t : G \to \mathbb{K}^\star$ is a section such that $f = dt : G \times G \to \mathbf{k}^\star$ is a factor set, we will say that t is a **section with factor set** $f = dt$.

2.14 Definition. Let G be an ordered Abelian group, and $\mathbf{k}$ a field be given. Given a 2 co-cycle $f : G \times G \to \mathbf{k}^\star$, the field of generalized power series $\mathbf{k}((G, f))$ with factor set f is the set of formal series $s = \sum_{\gamma \in G} a_\gamma t^\gamma$, with $a_\gamma \in \mathbf{k}$, whose support $\operatorname{supp} s := \{\gamma\ ;\ \gamma \in G \text{ and } a_\gamma \neq 0\}$ is a well-ordered subset of G. Sum and multiplication are defined formally, with the condition

$$t^\alpha t^\beta = f[\alpha, \beta] t^{\alpha+\beta}.$$

It is well-known that $\mathbf{k}((G, f))$ is a valued field, with valuation given by $v(s) := \min \operatorname{supp} s$ (by convention set $\min \operatorname{supp} s = \infty$ if $\operatorname{supp} s = \emptyset$), value group G, residue field $\mathbf{k}$ and canonical section $t(\gamma) := t^\gamma$. With this definition, t is a section with factor set f, and $\mathbf{k}(G, f)$ is the subfield of $\mathbf{k}((G, f))$ generated by $\mathbf{k} \cup \left\{ t^\gamma : \gamma \in G \right\}$. If $f = 1$ we denote $\mathbf{k}((G, f))$ by $\mathbf{k}((G))$.

A subfield F of $\mathbf{k}((G, f))$ is **truncation closed** if whenever $s = \sum_{\gamma \in G} a_\gamma t^\gamma \in F$ and $g \in G$, the restriction $s_{<g} = \sum_{\gamma \in G^{<g}} s_\gamma t^\gamma$ of s to the initial segment $G^{<g}$ of G also belongs to F. If F contains $\mathbf{k}(G, f)$, then F is truncation closed if and only if for all $s \in F$ we have $s_{<0} \in F$ (this is because $s_{<g} = (st^{-g})_{<0}$). Note that if F is truncation closed and $s \in F$, then $s_I := \sum_{\gamma \in I} s_\gamma t^\gamma$, the restriction of s to an initial segment I of G also belongs to F (this is because $\operatorname{supp} s$ is wellordered, so $s_I = s_{<g}$ for an appropriately chosen $g \in G$). Given a valued field $\mathbb{K}$ with residue field $\mathbf{k}$ and value group G with $\mathbf{k}(G, f) \subset \mathbb{K}$, a **truncation closed embedding** of $\mathbb{K}$ in $\mathbf{k}((G, f))$ over $\mathbf{k}(G, f)$, is an embedding φ such that φ is the identity on $\mathbf{k}(G, f)$ and $F := \varphi(\mathbb{K})$ is truncation closed. Note that since the restriction of φ to $\mathbf{k}(G)$ is the identity, φ is in particular an embedding of $\mathbf{k}$-vector spaces.

3 Tower of complements

From now on, we shall assume that $\mathbb{K}$ is a valued field (with same characteristic as its residue field) which admits a value group section and a residue field section. Fix once and for all a residue field section ι (we will assume that ι is the identity), and a value group section $t : G \to \mathbb{K}^\star$.

From now on, we fix $\mathbf{R}$ a $\mathbf{k}$-subalgebra of $\mathbb{K}$ containing $\mathbf{k}$ and the image of t. Given $\Lambda \in \breve{G}$, define

$$\mathcal{O}[\Lambda] := \left\{ x \in \mathbf{R} : v(x) \in \Lambda^R \right\} = \left\{ x \in \mathbf{R} : v(x) > \Lambda \right\}.$$

Note that these are precisely the (possibly fractional) ideals of the valuation ring.

3.1 Remark.

1. $\mathcal{O}[\Lambda]$ is a $\mathbf{k}$-linear subspace of $\mathbf{R}$;
2. for every $\gamma \in \Lambda^R$, $t^\gamma \in \mathcal{O}[\Lambda]$, in particular, $t^\gamma \in \mathcal{O}[\gamma^-]$;
3. $\mathcal{O}[0^-] = \mathcal{O}$;
4. $\mathcal{O}[0^+] = \mathcal{M} \cap \mathbf{R}$;
5. $\mathcal{O}[\gamma^\pm] = t^\gamma \mathcal{O}[0^\pm]$;
6. $\Gamma \leq \Lambda$ if and only if $\mathcal{O}[\Gamma] \supseteq \mathcal{O}[\Lambda]$;
7. if Λ^R has no minimum, then $\mathcal{O}[\Lambda] = \bigcup_{\gamma > \Lambda} \mathcal{O}[\gamma^\pm] = \bigcap_{\gamma < \Lambda} \mathcal{O}[\gamma^+]$;
8. $\mathcal{O}[\Lambda \mathbin{+^{\mathrm{R}}} \Gamma] = \mathcal{O}[\Lambda]\mathcal{O}[\Gamma]$;
9. $\mathcal{O}[\gamma + \Lambda] = t^\gamma \mathcal{O}[\Lambda]$.

3.2 Definition. A (t-compatible) **weak tower of complements** (of the valuation ring $\mathcal{O}$) for $\mathbf{R}$ is a family $\mathcal{A} = \left\{ A[\Lambda] : \Lambda \in \breve{G} \right\}$ of subsets of $\mathbf{R}$ indexed by $\breve{G}$, such that:

CA. $A[\Lambda]$ is a $\mathbf{k}$-subspace of $\mathbf{R}$;

CB. $A[\Lambda] \oplus \mathcal{O}[\Lambda] = \mathbf{R}$, as $\mathbf{k}$-spaces;

CC. $t^\gamma \mathbf{k} \subseteq A[\gamma^+]$;

CD. $\Gamma \leq \Lambda$ if and only if $A[\Gamma] \subseteq A[\Lambda]$;

CF. $A[\gamma+\Lambda] = t^\gamma A[\Lambda]$

In particular, $\mathbf{A} := A[0^-]$ is a $\mathbf{k}$-complement of $\mathcal{O}\cap\mathbf{R}$. Note that $A[+\infty] = \mathbf{R}$, and $A[-\infty] = \{0\}$.

A (t-compatible) **tower of complements** is a (t-compatible) weak tower of complements satisfying the following additional axiom, instead of Axiom **C**F:

CE. $A[\Lambda + \Gamma] \supseteq A[\Lambda]A[\Gamma]$.

We then also say that $\mathcal{A}$ **is multiplicative**. In this case, $\mathbf{A}$ is a $\mathbf{k}$-algebra complement.

3.3 Remark. Any tower of complements is a weak tower of complements.[3]

Proof. We have to prove that if $\mathcal{A}$ is a tower of complements, then $t^\gamma A[\Lambda] = A[\gamma+\Lambda]$. Since $t^\gamma \in A[\gamma^+]$, we have $t^\gamma A[\Lambda] \subseteq A[\gamma^+ + \Lambda] = A[\gamma+\Lambda]$. Conversely, using the above inclusion we get:
$A[\gamma+\Lambda] = t^\gamma t^{-\gamma} A[\gamma+\Lambda] \subseteq t^\gamma A[-\gamma+\gamma+\Lambda] = t^\gamma A[\Lambda]$. $\square$

For the remainder of this section, we will assume that $\mathcal{A}$ is a weak tower of complements for $\mathbf{R}$, unless we explicitly specify otherwise. In the following Lemma, we list useful properties of weak towers of complements.

3.4 Lemma. *1.* $\mathbf{k} \subseteq A[0^+]$.

2. $A[\gamma^-] = t^\gamma \mathbf{A}$.

3. If $\Gamma \leq \Lambda$, then

$$\begin{aligned} \mathbf{R} &= A[\Gamma] \oplus \big(\mathcal{O}[\Gamma]\cap A[\Lambda]\big) \oplus \mathcal{O}[\Lambda], \\ A[\Lambda] &= A[\Gamma] \oplus \big(\mathcal{O}[\Gamma]\cap A[\Lambda]\big), \qquad \text{and} \\ \mathcal{O}[\Gamma] &= \big(\mathcal{O}[\Gamma]\cap A[\Lambda]\big) \oplus \mathcal{O}[\Lambda]. \end{aligned}$$

4. Let $\Gamma \leq \Lambda$. Let $x = x_1 + x_2$, with $x_1 \in A[\Gamma]$ and $x_2 \in \mathcal{O}[\Gamma]$; if $x \in A[\Lambda]$, then $x_i \in A[\Lambda]$; if $x \in \mathcal{O}[\Lambda]$, then $x_i \in \mathcal{O}[\Lambda]$.

5. $\mathcal{O}[\gamma^-] \cap A[\gamma^+] = t^\gamma \mathbf{k}$.

6. $A[\gamma^+] = A[\gamma^-] \oplus t^\gamma \mathbf{k}$.

[3]Example 5.26 produces a W.T.o.C. that is not a T.o.C..

7. If Λ^R has no minimum, then $A[\Lambda] = \bigcap_{\gamma \in \Lambda^R} A[\gamma^\pm]$.

Proof. 1. This follows immediately from $t^0 = 1$ and Axiom **C**C.

2. We have $t^\gamma \in A[\gamma^+]$, so $t^\gamma \mathbf{A} \subseteq A[\gamma^+]\mathbf{A} \subseteq A[\gamma^+ + 0^-] = A[\gamma^-]$. Conversely, let $x \in A[\gamma^-]$, $y := t^{-\gamma}x$. Decompose $y = y_1 + y_2$, $y_1 \in \mathbf{A}$, $y_2 \in \mathcal{O}$. Hence, $x = t^\gamma y = t^\gamma y_1 + t^\gamma y_2$. We have that $t^\gamma y_1 \in A[\gamma^-]$ by the previous point. Moreover, $v(t^\gamma y_2) = \gamma + v(y_2) \geq \gamma$, therefore $t^\gamma y_2 \in \mathcal{O}[\gamma^-]$. However, $x \in A[\gamma^-]$, hence $t^\gamma y_2 \in \mathcal{O}[\gamma^-] \cap A[\gamma^-] = 0$, so $x = t^\gamma y_1 \in t^\gamma \mathbf{A}$.

3. Let $x \in \mathbf{R}$. Write $x = x_1 + y$, where $x_1 \in A[\Gamma]$ and $y \in \mathcal{O}[\Gamma]$. Write $y = x_2 + x_3$, where $x_2 \in A[\Lambda]$ and $x_3 \in \mathcal{O}[\Lambda]$. Note that $x_3 \in \mathcal{O}[\Gamma]$, hence $x_2 = y - x_3 \in \mathcal{O}[\Gamma]$. Therefore, we have decomposed x as $x = x_1 + x_2 + x_3$.
To prove the uniqueness of the decomposition, assume that $x = x_1' + x_2' + x_3'$, with $x_1' \in A[\Gamma]$, $x_2' \in \mathcal{O}[\Gamma] \cap A[\Lambda]$, and $x_3' \in \mathcal{O}[\Lambda]$. Define $y' := x_2' + x_3'$. Note that $x_3' \in \mathcal{O}[\Gamma]$, and therefore $y \in \mathcal{O}[\Gamma]$. By Axiom **C**B, $x_1' = x_1$, and therefore $y' = y$. Again by Axiom **C**B, $x_2' = x_2$ and $x_3' = x_3$.
The other assertions follow by similar considerations.

4. Immediate from the previous point.

5. It is immediate by **C**C that $\mathcal{O}[\gamma^-] \cap A[\gamma^+] \supset t^\gamma \mathbf{k}$. Conversely, let $x \in \mathcal{O}[\gamma^-] \cap A[\gamma^+]$, $x \neq 0$. Hence, $v(x) = \gamma$ by **C**B. Write x as $ct^\gamma + z$, with $c \in \mathbf{k}^\star$, and $z \in \mathcal{O}[\gamma^+]$. By **C**C, $ct^\gamma \in A[\gamma^+]$, hence $z \in A[\gamma^+] \cap \mathcal{O}[\gamma^+]$, therefore $z = 0$.

6. $A[\gamma^+] = A[\gamma^-] \oplus \big(A[\gamma^+] \cap \mathcal{O}[\gamma^-]\big) = A[\gamma^-] \oplus t^\gamma \mathbf{k}$.

7. By **C**D we have $A[\Lambda] \subseteq \bigcap_{\gamma \in \Lambda^R} A[\gamma^\pm]$. Conversely, since Λ^R has no minimum, $\bigcap_{\gamma \in \Lambda^R} A[\gamma^+] = \bigcap_{\gamma \in \Lambda^R} A[\gamma^-]$. Let $x \in \bigcap_{\gamma \in \Lambda^R} A[\gamma^+]$. Decompose $x = x_1 + x_2$, $x_1 \in A[\Lambda]$, $x_2 \in \mathcal{O}[\Lambda]$. By the previous inclusion, $x_1 \in \bigcap_{\gamma \in \Lambda^R} A[\gamma^-]$. Assume for a contradiction that $x_2 \neq 0$. Then $\gamma := v(x_2) > \Lambda$. Since Λ^R has no minimum, $\Lambda < \gamma^-$, therefore $x \in A[\gamma^-]$, and thus $x_2 \in A[\gamma^-]$. Hence, $x_2 \in A[\gamma^-] \cap \mathcal{O}[\gamma^-] = 0$, a contradiction. □

3.5 Corollary. *The weak tower of complements $\mathcal{A}$ is uniquely determined by $\mathbf{A} = A[0^-] \in \mathcal{A}$.*

Proof. First observe that by 2. and 6. of Lemma 3.4 $A[0^-]$ uniquely determines $A[\gamma^\pm]$ for all $\gamma \in G$. Second, if Γ is a cut of G not of the form $\gamma^\pm$, then $A[\Gamma]$ is the intersection of all $A[\gamma^\pm]$ for $\Gamma < \gamma \in G$ (Lemma 3.4 7.). □

3.6 Corollary. *t is a section with factor set $dt : G \times G \to \mathbf{k}^\star$.*

Proof. We compute:

$$d := t^\alpha t^\beta t^{-\alpha-\beta} \in t^\alpha t^\beta t^{-\alpha-\beta} A[0^+] = A[\alpha + \beta + (-\alpha - \beta) + 0^+] = A[0^+].$$

Hence, $d \in \mathcal{O} \cap A[0^+] = \mathbf{k}$. □

3.7 Main example. Let $f : G \times G \to \mathbf{k}^\star$ be a factor set. For every $\Lambda \in \breve{G}$, define $\mathbf{k}((\Lambda)) := \mathbf{k}[[\Lambda^L, f]]$ as the subset of $\mathbf{k}((G, f))$ of the power series with support contained in Λ^L. The family $\{ \mathbf{k}((\Lambda)) : \Lambda \in \breve{G} \}$ is a tower of complements for $\mathbf{k}((G, f))$.
Moreover, if t has factor set $f = dt$, and φ is a truncation-closed embedding of $\mathbb{K}$ in $\mathbf{k}((G, f))$, preserving the section t and the embedding of the residue field $\mathbf{k}$, then the family

$$\mathcal{A} := \left\{ \varphi^{-1}\big(\mathbf{k}((\Lambda))\big) : \Lambda \in \breve{G} \right\}$$

is a tower of complements for $\mathbb{K}$, compatible with t.

3.8 Remark. Note that in this case $\bigcup_{\substack{\Lambda \in \breve{G} \\ \Lambda < +\infty}} A[\Lambda] \subsetneq \mathbb{K}$. Since $\mathbb{K} = A[+\infty]$, we therefore have at least one cut ($\Gamma = +\infty$) such that $A[\Gamma]$ is not equal to the union of $A[\Lambda]$, for $\Lambda < \Gamma$, because the image of $\mathbb{K}$ will always contain power series with unbounded support. This shows that $\mathcal{A}$ is *not* a "continuous family": if Γ is a supremum of an increasing sequence of cuts Λ_i, then $A[\Gamma]$ is *not necessarily* the union of the $A[\Lambda_i]$'s. Note that if instead Γ is an infimum of a decreasing sequence of cuts Λ_i, then $A[\Gamma]$ is the intersection of the $A[\Lambda_i]$'s (Lemma 3.4 7.).

3.1 Embeddings in power series

3.9 Definition. For every $\gamma \in G$, we can write

$$\mathbf{R} = A[\gamma^-] \oplus t^\gamma \mathbf{k} \oplus \mathcal{O}[\gamma^+] = t^\gamma \mathbf{A} \oplus t^\gamma \mathbf{k} \oplus t^\gamma \mathcal{M} \cap \mathbf{R}.$$

Given $x \in \mathbf{R}$, x decomposes uniquely as $x = x_1 + a_\gamma t^\gamma + x_3$, where $x_1 \in A[\gamma^-]$, $a_\gamma \in \mathbf{k}$ and $x_3 \in \mathcal{O}[\gamma^+]$. Consider the formal sum $\mathcal{S}x := \sum_{\gamma \in G} a_\gamma t^\gamma$. $\mathcal{S}$ defines a map from $\mathbf{R}$ to ${}^G\mathbf{k}$, the set of maps from G to $\mathbf{k}$. Note that ${}^G\mathbf{k}$ is an Abelian group under point-wise addition; it is obvious that $\mathcal{S}$ is a homomorphism of (additive) groups. Define also $\operatorname{supp} x := \operatorname{supp} \mathcal{S}x$.

Observe that the definitions of $\mathcal{S}x$ and supp depend on the given tower of complements $\mathcal{A}$.

3.10 Lemma. *$\mathcal{S}$ is injective. Moreover, $v(x) = \min(\operatorname{supp} x)$.*

Proof. Let $x \in \mathbf{R}$, and $\gamma := v(x)$. Note that if $\gamma < \infty$, then $a_\gamma \neq 0$. Therefore, $\gamma \in \operatorname{supp} x$. Hence, if $\mathcal{S}x = 0$, then $\gamma = \infty$, that is, $x = 0$, and $\mathcal{S}$ is injective. Conversely, let $\lambda < \gamma$, and assume for contradiction that $a_\lambda \neq 0$. Then, $x = x_1 + a_\lambda t^\lambda + x_3$, where $x_1 \in A[\lambda^-]$ and $x_3 \in \mathcal{O}[\lambda^+]$. Therefore, $\gamma = v(x) = \min\{v(x_1), \lambda, v(x_3)\} \leq \lambda$, a contradiction. □

We now prove that the image of $\mathcal{S}$ is contained in $\mathbf{k}((G))$.

3.11 Proposition. *If $x \in \mathbf{R}$, then $\operatorname{supp} \mathcal{S}x$ is well-ordered.*

Proof. Suppose not. Let $\alpha := vx$. Let $\gamma_1 > \gamma_2 > \ldots \in \operatorname{supp} \mathcal{S}x$. Since $\operatorname{supp} \mathcal{S}x$ is bounded below by α, and $\breve{G}$ is complete, there exists

$$\Lambda := \inf\left\{ \gamma_i^+ : i \in \mathbb{N} \right\} = \inf\left\{ \gamma_i^- : i \in \mathbb{N} \right\} \;= \left\{ \gamma_i : i \in \mathbb{N} \right\}^- \in \breve{G}.$$

Moreover, Λ^R has no minimum, and $\alpha < \Lambda$. Decompose $x = x_1 + x_2$, $x_1 \in A[\Lambda]$, $x_2 \in \mathcal{O}[\Lambda]$. Let $\beta := v(x_2) > \alpha$, and let $\gamma := \gamma_i$ such that $\gamma < \beta$ (it exists, because Λ^R has no minimum). Write $x = y_1 + a_\gamma t^\gamma + y_2$, $y_1 \in A[\gamma^-]$, $v(y_2) > \gamma$, and $a_\gamma \neq 0$. However, $x_1 \in A[\Lambda] \subseteq A[\gamma^-]$, and $x_2 \in \mathcal{O}[\beta^-] \subseteq \mathcal{O}[\gamma^-]$, therefore, by **C**B, $x_1 = y_1$, and $x_2 = a_\gamma t^\gamma + y_2$, a contradiction to $v(x_2) = \beta > \gamma$. □

Since the minimum of $\operatorname{supp} \mathcal{S}x$ is exactly vx, we have that:

3.12 Corollary. *The map $\mathcal{S}$ is an embedding of (additive) valued groups.*

For the rest of this section, we will assume that $\mathcal{A}$ is a (weak) tower of complements for $\mathbf{R}$.

3.13 Definition. Set: $\mu(x) := (\operatorname{supp} x)^+ \in \breve{G}$.

Note that $\mu(x) = -\infty$ if and only if $x = 0$.

3.14 Lemma. *Let $\Lambda \in \breve{G}$ and $x, y \in \mathbf{R}$.*

1. *If $\mu(x) = \Lambda$, then $x \in A[\Lambda]$.*
2. *In general, $x \in A[\Lambda]$ if and only if $\Lambda \geq \mu x$. Equivalently, $A[\Lambda] = \left\{ x : \mu x \leq \Lambda \right\}$.*

3. $\mu x = \inf\{\, \Lambda : x \in A[\Lambda] \,\}$.

4. $\mu(x+y) \leq \max(\mu x, \mu y)$.

5. If $\mathcal{A}$ *is a* tower of complements*, then* $\mu(xy) \leq \mu x + \mu y$.

Proof. 1. Decompose $x = x_1 + x_2$, $x_1 \in A[\Lambda]$, $x_2 \in \mathcal{O}[\Lambda]$. If $x_2 \neq 0$, then $v(x_2) \in \Lambda^R$. However, $v(x_2) \in \operatorname{supp} x$, a contradiction.

2. $\Leftarrow$) By 3.14–1 and **C**D.
$\Rightarrow$) If $x \in A[\Lambda]$ and $\Lambda < \mu x$, then there exists $\gamma \in \operatorname{supp} x$ such that $\gamma > \Lambda$. Decompose $x = x_1 + c_\gamma t^\gamma + x_2$. Decompose $x_1 = z_1 + z_2$, $z_1 \in A[\Lambda]$, $z_2 \in \mathcal{O}[\Lambda] \cap A[\gamma^-]$. Hence, $x = z_1 + z_2 + c_\gamma t^\gamma + x_2$. Since $x \in A[\Lambda]$, $z_2 + c_\gamma t^\gamma + x_2 = 0$. Thus, $z_2 \in A[\gamma^-] \cap \mathcal{O}[\gamma^-] = (0)$, so $c_\gamma = 0$, contradicting the fact that $\gamma \in \operatorname{supp} x$.

3. This is a rewording of 3.14–2.

4. This follows from Axiom **C**A, plus 3.14–2.

5. If either $x = 0$ or $y = 0$, the conclusion is trivial. By 3.14–1, $x \in A[\mu x]$, $y \in A[\mu y]$, therefore $xy \in A[\mu x + \mu y]$ by **C**E. This, together with 3.14–2, implies that $\mu(xy) \leq \mu x + \mu y$. □

Assume now that $\mathbf{R} = \mathbb{K}$ and that $\mathcal{A}$ is multiplicative. Let $f = dt$. We show that $\mathcal{S}$ is an embedding of valued fields in the power series field, with multiplication twisted by dt.

3.15 Theorem. *For every* $x, y \in \mathbb{K}$*,* $\mathcal{S}(xy) = (\mathcal{S}x)(\mathcal{S}y)$ *(with the multiplication of* $\mathbf{k}((G,f))$*). Therefore,* $\mathcal{S}$ *is a truncation-closed embedding of valued fields of* $\mathbb{K}$ *in* $\mathbf{k}((G,f))$*.*

Proof. If not, let $x, y \in \mathbb{K}$ of minimal length[4] such that $\mathcal{S}(xy) \neq (\mathcal{S}x)(\mathcal{S}y)$. Let $z := \mathcal{S}x\mathcal{S}y$, and $\gamma \in G$ minimal with $\mathcal{S}(xy)(\gamma) \neq z(\gamma)$.[5] Since $\mu(xy) \leq \mu x + \mu y$ and $\mu(\mathcal{S}x\mathcal{S}y) \leq \mu(\mathcal{S}x) + \mu(\mathcal{S}y) = \mu x + \mu y$, $\gamma < \mu x + \mu y$. Let $\alpha, \beta \in G$ such that $\alpha < \mu x$, $\beta < \mu y$ and $\alpha + \beta = \gamma$. Decompose $x = x_1 + a_\alpha t^\alpha + x_2$, $y = y_1 + b_\beta t^\beta + y_2$. Thus,

$$\begin{aligned} xy &= (x_1 + a_\alpha t^\alpha + x_2)(y_1 + b_\beta t^\beta + y_2) \\ &= x_1 y + (a_\alpha t^\alpha + x_2) y_1 + f[\alpha,\beta] a_\alpha b_\beta t^\gamma + \mathrm{o}(t^\gamma) \end{aligned}$$

[4] The length of x is the order type of $\operatorname{supp} \mathcal{S}x$.
[5] We are using the notation $(\sum a_\lambda t^\lambda)(\gamma) := a_\gamma$.

Here $o(t^\gamma)$ denotes an element of value $> \gamma$. Since $\alpha < \mu x$, $a_\alpha t^\alpha + x_2 \neq 0$, and similarly $b_\beta t^\beta + y_2 \neq 0$. By minimality of x and y, $\mathcal{S}(x_1 y) = \mathcal{S}x_1 \mathcal{S}y$ and $\mathcal{S}\big((a_\alpha t^\alpha + x_2)y_1\big) = \mathcal{S}(a_\alpha t^\alpha + x_2)\mathcal{S}y_1$. Moreover, $\mathcal{S}\big(f[\alpha,\beta]a_\alpha b_\beta t^\gamma\big) = f[\alpha,\beta]a_\alpha b_\beta t^\gamma$ by definition of $\mathcal{S}$, and $\mathcal{S}\big(o(t^\gamma)\big) = o(t^\gamma)$. Therefore,

$$\begin{aligned}\mathcal{S}(xy) &= \mathcal{S}(x_1 y) + \mathcal{S}\big((a_\alpha t^\alpha + x_2)y_1\big) + \mathcal{S}\big(f[\alpha,\beta]a_\alpha b_\beta t^\gamma\big) + \mathcal{S}(o(t^\gamma))\\ &= \mathcal{S}x_1\mathcal{S}y + \mathcal{S}(a_\alpha t^\alpha + x_2)\mathcal{S}y_1 + a_\alpha t^\alpha b_\beta t^\beta + o(t^\gamma)\\ &= \mathcal{S}x\mathcal{S}y - a_\alpha t^\alpha \mathcal{S}y_2 - \mathcal{S}x_2(b_\beta t^\beta + \mathcal{S}y_2) + o(t^\gamma).\end{aligned}$$

Thus, $z - \mathcal{S}(xy) = o(t^\gamma)$, a contradiction. Finally observe that, by the uniqueness of the decomposition used in the definition of $\mathcal{S}$, $\mathcal{S}$ is truncation closed. □

3.16 Corollary. *Let $f = dt$ be the factor set of t. There is a one-to-one correspondence between towers of complements for $\mathbb{K}$ and truncation-closed embeddings of $\mathbb{K}$ in $\mathbf{k}((G, f))$.*

Proof. Consider the maps $\varphi \mapsto A_\phi$ that maps a truncation-closed embedding φ to the corresponding T.o.C. (induced by φ), and $\mathcal{A} \mapsto \mathcal{S}_\mathcal{A}$ that maps a T.o.C. $\mathcal{A}$ to the truncation-closed embedding induced by it. We want to prove that the two maps are inverses of each other. We know that a T.o.C. $\mathcal{A}$ is uniquely determined by the corresponding function $\mu_\mathcal{A}$. Let $\mathcal{B}$ be a T.o.C. Let us prove that $\mathcal{A}_{\mathcal{S}_\mathcal{B}} = \mathcal{B}$. For a cut Γ, we have that

$$x \in B[\Gamma] \iff \mathcal{S}_\mathcal{B}(x) \in \mathbf{k}((\Gamma)) \iff x \in \mathcal{A}_{\mathcal{S}_\mathcal{B}}[\Gamma]\,.$$

Conversely, let φ be a truncation-closed embedding. Let us prove that $\alpha := \mathcal{S}_{A_\varphi} = \varphi$. Let $\mathcal{B} := \mathcal{A}_\varphi$. Assume, for a contradiction, that φ is not equal to α, and choose $x \in \mathbb{K}$ of minimal length (w.r.t. the T.o.C. $\mathcal{B}$) such that $\varphi(x)$ is not equal to $\alpha(x)$. Note that

$$\varphi(x) \in \mathbf{k}((\Gamma)) \iff x \in B[\Gamma] \iff \alpha(x) \in \mathbf{k}((\Gamma))\,. \tag{3.1}$$

Let $\Gamma < \mu_\mathcal{B}(x)$; split $x = x_1 + x_2$ at Γ. By the minimality of x, $\varphi(x_1) = \alpha(x_1)$. Therefore, $v(\varphi(x) - \alpha(x)) > \Gamma$ for every $\Gamma < \mu_B(x)$, and thus $v(\varphi(x) - \alpha(x)) > \mu(x)$, contradicting (3.1). □

4 Truncation k-algebra complements of $\mathcal{O}$

Fix a residue field section $\iota : \mathbf{k} \to \mathbb{K}$, which we assume to be the inclusion map. Let $t : \mathbb{K}^\star \to G$ be a value group section, with factor set $f = dt$. A $\mathbf{k}$-complement $\mathbf{A}$ of $\mathcal{O}$ is **compatible with** t if $t^\gamma \in \mathbf{A}$ for every $0 > \gamma \in G$.

Recall that we call a $\mathbf{k}$-algebra complement $\mathbf{A}$ of $\mathcal{O}$ a truncation $\mathbf{k}$-algebra complement if there is a truncation-closed embedding $\varphi : \mathbb{K} \to \mathbf{k}((G, f'))$ (preserving ι) for some factor set f', such that $\mathbf{A} = \varphi^{-1}\big(\mathbf{k}((G^{<0}, f'))\big)$. We say that $\mathbf{A}$ is a **truncation k-algebra complement compatible with** t if moreover $\varphi(t^\gamma) = t^\gamma$ for every $\gamma \in G$, and $f = f'$.

It follows from Corollaries 3.5 and 3.16 that $\mathbf{A}$ is a truncation $\mathbf{k}$-algebra complement compatible with t if and only if $\mathbf{A} = A[0^-]$, for some T.o.C. $\mathcal{A}$ compatible with t.

Our aim in this section is to find a valued field $\mathbb{K}$, with a residue field section ι and a value group section t, and a $\mathbf{k}$-algebra complement $\mathbf{A}$ compatible with t, such that $\mathbf{A}$ is not a truncation $\mathbf{k}$-algebra complement compatible with t. We will leave open the question whether $\mathbf{A}$ might be a truncation $\mathbf{k}$-algebra complement compatible with a different value group section t'.

Definition 1. Let $\mathcal{A} := \big(A[\Gamma]\big)_{\Gamma \in \breve{G}}$ be a family of subsets of $\mathbb{K}$, indexed by $\breve{G}$. $\mathcal{A}$ is a t-compatible **candidate weak tower of complements** (C.W.T.o.C. for short) if it satisfies the axioms CA, CC, CD, CF, and instead of CB the following axioms:

CB1. $A[\Lambda] \cap \mathcal{O}[\Lambda] = 0$;

CB2. $A[0^-] + \mathcal{O}[0^-] = \mathbb{K}$;

CG. $A[\Gamma] = \bigcup_{\lambda > \Gamma} A[\lambda^-]$.

If in addition $\mathcal{A}$ satisfies the axiom CE, then we say that $\mathcal{A}$ is multiplicative, or that $\mathcal{A}$ is a t-compatible **candidate tower of complements** (C.T.o.C. for short).

4.1 Remark. Every W.T.o.C. is a C.W.T.o.C.; every T.o.C. is a C.T.o.C..

4.2 Lemma. *Let $\mathcal{A}$ be a C.W.T.o.C.. Then,*

1. $A[\gamma^-] = t^\gamma A[0^-]$*;*

2. $A[\gamma^+] = A[\gamma^-] + t^\gamma \mathbf{k} = t^\gamma A[0^+]$;

3. $\mathbb{K} = A[\gamma^-] \oplus \mathcal{O}[\gamma^-] = A[\gamma^+] \oplus \mathcal{O}[\gamma^+] = A[\gamma^-] \oplus t^\gamma \mathbf{k} \oplus \mathcal{O}[\gamma^+]$.

Given a C.W.T.o.C. $\mathcal{A}$, $x \in \mathbb{K}$ and $\lambda \in G$, we can decompose $x = x'_\lambda + a_\lambda(x) t^\lambda + x''_\lambda$ uniquely, in such a way that $x'_\lambda \in A[\lambda^-]$, $a_\lambda(x) \in \mathbf{k}$, and $v(x''_\lambda) > \lambda$. We define $\mathcal{S}x := \sum_\lambda a_\lambda(x) t^\lambda$ and $\operatorname{supp} x := \operatorname{supp} \mathcal{S}x$ as in Section 3.1.

The proofs of the following two lemmata are easy.

4.3 Lemma. *Let $\mathcal{A}$ be a C.W.T.o.C.. The following are equivalent:*

1. *$\mathcal{A}$ is a W.T.o.C.;*

2. *for every $\Gamma \in \breve{G}$, $A[\Gamma] + \mathcal{O}[\Gamma] = \mathbb{K}$;*

3. *for every $x \in \mathbb{K}$, $\operatorname{supp} x$ is well-ordered;*

4. *for every $x \in \mathbb{K}$, for every $\Gamma \in \breve{G}$, there exists $\bar{\lambda} > \Gamma$ such that, for every $\Gamma < \lambda < \bar{\lambda}$, $x'_\lambda = x'_{\bar{\lambda}}$;*

5. *for every $x \in \mathbb{K}$, for every $\Gamma \in \breve{G}$, there exists $\bar{\lambda} > \Gamma$, such that $x'_{\bar{\lambda}} \in A[\Gamma]$.*

4.4 Lemma. *Let A^- be a $\mathbf{k}$-complement of $\mathcal{O}$ compatible with t. Then there exists a unique C.W.T.o.C. $\mathcal{A}$ such that $A[0^-] = A^-$. $\mathcal{A}$ is multiplicative iff A^- is multiplicative.*
$\mathcal{A}$ is defined in the following way:

$$A[\Lambda] := \bigcap_{\gamma > \Lambda} t^\gamma A^-.$$

Let A^- be a $\mathbf{k}$-complement of $\mathcal{O}$ compatible with t, and the family $\mathcal{A}$ defined in the lemma be the C.W.T.o.C. induced by A^-.

If A^- is a truncation $\mathbf{k}$-algebra compatible with t, then there exist at least one W.T.o.C. $\mathcal{A}'$ such that $\mathcal{A}'[0^-] = A^-$. Then by the above lemma, $\mathcal{A}' = \mathcal{A}$. Therefore, a necessary and sufficient condition for A^- to be a truncation $\mathbf{k}$-algebra compatible with t is that, for every $x \in \mathbb{K}$, $\operatorname{supp} x$ is well-ordered.

We will now define a valued field $\mathbb{K}$, a $\mathbf{k}$-complement B^- of $\mathcal{O}$ (compatible with chosen residue field and value group sections), and an element $d \in \mathbb{K}$, such that $\operatorname{supp} d$ is not well-ordered.

Fix a field $\mathbf{f}$. Let $\mathbb{F} := \mathbf{f}((\mathbb{Z}))$, with the canonical inclusion $\iota : \mathbf{f} \to \mathbb{F}$ and value group section $t : \mathbb{Z} \to \mathbb{F}$. Call $\rho : \mathcal{O} \to \mathbf{f}$ the residue map. Let $A^- := \mathbf{f}[[\mathbb{Z}^{<0}]]$: by definition, A^- is a truncation $\mathbf{f}$-algebra. Define the maps $h', h'' : \mathbb{F} \times \mathbb{Z} \to \mathbb{F}$ and $g : \mathbb{F} \times \mathbb{Z} \to \mathbf{f}$, $h'(x, \gamma) = x'_\gamma$, $g(x, \gamma) = a_\gamma(x)$, $h''(x, \gamma) = x''_\gamma$. Let $c := \sum_{n \geq 0} t^n \in \mathbb{F}$. Consider the first-order structure, in the sorts $\mathbb{F}, \mathbf{f}, \mathbb{Z}$,

$$M := \big(\mathbb{F}, \mathbf{f}, \mathbb{Z}; A^-, +_{\mathbb{F}}, \cdot_{\mathbb{F}}, +_{\mathbb{Z}}, \leq_{\mathbb{Z}}, +_{\mathbf{f}}, \cdot_{\mathbf{f}}, v, t, \rho, \iota, h', g, h'', c\big).$$

Let

$$\tilde{M} = \big(\mathbb{K}, \mathbf{k}, G; B^-, +_{\mathbb{K}}, \cdot_{\mathbb{K}}, +_G, \leq_G, +_{\mathbf{k}}, \cdot_{\mathbf{k}}, \tilde{v}, \tilde{t}, \tilde{\rho}, \tilde{\iota}, \tilde{h}', \tilde{g}, \tilde{h}'', c\big)$$

be an ω-saturated elementary extension of M. It is easy to see that B^- is an $\mathbf{f}$-complement to $\mathcal{O}$, satisfying $B^- \cdot B^- \subseteq B^-$, and that $\tilde{h}'(x, \gamma) = x'_\gamma$, $\tilde{g}(x, \gamma) = a_\gamma(x)$, $\tilde{h}''(x, \gamma) = x''_\gamma$. Moreover, since $g(c, \gamma) = 1$ for every $\gamma \in \mathbb{Z}^{\geq 0}$ holds in M, we must have that $\tilde{g}(c, \gamma) = 1$ for every $\gamma \in G^{\geq 0}$. Thus, $\operatorname{supp} c = G^{\geq 0}$. However, since $\tilde{M}$ is ω-saturated, $G^{\geq 0}$ is not well-ordered. Therefore, B^- is not a truncation $\mathbf{k}$-algebra compatible with $\tilde{t}$.

5 Building a tower of complements

Let $\mathbb{K}$, G and $\mathbf{k}$ be as in §3. To simplify the notation, we will assume that the section $t : G \to \mathbb{K}^\star$ has trivial factor set. The aim of this section is to build a tower of complements for $\mathbb{K}$. Given $\mathbf{R}$ a subring of $\mathbb{K}$ containing $\mathbf{k}$ and the image of t, we define a tower of complements for $\mathbf{R}$ as a family $\mathcal{A}$ of subsets of $\mathbf{R}$ satisfying the axioms **C**A–**C**E.

5.1 The basic case

First of all, consider $\mathbf{k}[G]$, the subring of $\mathbb{K}$ generated by the monomials ct^γ. There is one and only one tower of complements for $\mathbf{k}[G]$: for each Λ, $A[\Lambda]$ is the $\mathbf{k}$-vector subspace of $\mathbb{K}$ generated by the monomials t^γ such that $\gamma < \Lambda$.

5.2 Extension to quotient fields

Next, consider $\mathbf{k}(G)$, the field of quotients of $\mathbf{k}[G]$. There is one quick way of constructing a tower of complements for $\mathbf{k}(G)$: notice that there exists a unique analytic embedding φ from $\mathbf{k}(G)$ in the field of power series $\mathbf{k}((G))$

preserving $\mathbf{k}$ and the section t. Moreover, φ is truncation-closed, hence the family

$$\mathcal{A} := \{ \varphi^{-1}(\mathbf{k}((\Lambda))) : \Lambda \in \breve{G} \}$$

is a tower of complements for $\mathbf{k}(G)$.

However, as we intend to construct our complements intrinsically, without the use of truncation-closed embeddings in power series fields, we wish to give a general construction for the extension of towers of complements from a ring to its quotient field.

5.1 Definition. Given $\Lambda \in \breve{G}$, define $\mathbb{Z}\Lambda := \sup\{ n\Lambda : n \in \mathbb{Z} \} \in \breve{G}$.

Note that $\mathbb{Z}\Lambda + \mathbb{Z}\Lambda = \mathbb{Z}\Lambda > 0$.

5.2 Proposition. *Let* $\mathbf{R}$ *a subring of* $\mathbb{K}$ *containing* $\mathcal{O}$ *and the image of* t. *Suppose that* $\mathcal{A}$ *is a tower of complements for* $\mathbf{R}$. *Define a tower* $\mathcal{B}$ *on the quotient field of* $\mathbf{R}$ *by*

$$B[\Gamma] = \operatorname{span}_k \Big\{ \frac{r}{1+a} : a, r \in R, v(a) > 0, \mu(r) + \mathbb{Z}\mu(a) \leq \Gamma \Big\}$$

for each $\Gamma \in \breve{G}$. *Then* $\mathcal{B}$ *is a tower of complements for the quotient field of* $\mathbf{R}$.

Proof. Property $\mathbf{C}$A for $\mathcal{B}$ holds by definition. Properties $\mathbf{C}$C and $\mathbf{C}$D for $\mathcal{B}$ are directly inherited from the corresponding properties of $\mathcal{A}$.

We show that $\mathcal{B}$ has property $\mathbf{C}$E. Take cuts Λ, Γ and elements $a, a', r, r' \in R$ with $v(a) > 0$, $v(a') > 0$, $\mu(r) + \mathbb{Z}\mu(a) \leq \Lambda$ and $\mu(r') + \mathbb{Z}\mu(a') \leq \Gamma$. By parts 3.14–4 and 3.14–5 of Lemma 3.14 we have that $\mu(rr') \leq \mu(r) + \mu(r')$ and $\mu(a + a' + aa') \leq \max(\mu(a), \mu(a'), \mu(a) + \mu(a')) = \mu(a) + \mu(a')$. It follows that $\mu(rr') + \mathbb{Z}\mu(a + a' + aa') \leq \mu(r) + \mathbb{Z}\mu(a) + \mu(r') + \mathbb{Z}\mu(a') \leq \Lambda + \Gamma$. Hence,

$$\frac{r}{1+a} \cdot \frac{r'}{1+a'} = \frac{rr'}{1+a+a'+aa'} \in B[\Lambda + \Gamma] .$$

By additivity, it follows that property $\mathbf{C}$E holds for $\mathcal{B}$.

Let us show that property $\mathbf{C}$B holds for $\mathcal{B}$. First, we prove that, for every cut Γ, $K = B[\Gamma] + \mathcal{O}[\Gamma]$. We will prove, by induction on the length of c, that for every $d \in R$, $0 \neq c \in R$, $n \in \mathbb{N}$ and Γ a cut of G, d/c^n splits at Γ.

W.l.o.g., $v(c) = v(d) = 0$, and $c = 1 - a$, for some $a \in R$ with $v(a) > 0$. If $a = 0$, then $d/c^n = d \in R$, and we are done. Otherwise, let $\Theta := \mu(c) = \mu(a)$. Note that $\Theta > 0$.

Split $d = d_1 + d_2$ at Γ. Note that $v(d_2/c^n) = v(d_2) > \Gamma$. Thus, it suffices to split d_1/c^n at Γ, and therefore, w.l.o.g., we can assume that $d = d_1$, and thus $\mu(d) \leq \Gamma$. If $d = 0$ we are done, otherwise $\Gamma > 0$, and therefore $\Gamma \geq \hat{\Gamma}$.

There are 4 cases: either $\mathbb{Z}\Theta \leq \hat{\Gamma}$, or $\hat{\Gamma} < \mathbb{Z}\Theta < \Gamma$, or $\mathbb{Z}\Theta = \Gamma$, or $\mathbb{Z}\Theta > \Gamma$.

If $\mathbb{Z}\Theta \leq \hat{\Gamma}$, then

$$\mu(d) + \mathbb{Z}\mu(c^n) \leq \mu(d) + \mathbb{Z}\mu(c) \leq \Gamma + \mathbb{Z}\Theta = \Gamma,$$

and therefore $d/c^n \in A[\Gamma]$.

If $\mathbb{Z}\Theta = \Gamma$, then $\Gamma = \hat{\Gamma}$, and we are in the previous case.

If $\mathbb{Z}\Theta > \Gamma$, we have $\Gamma < n_0\theta_0$ for some $n_0 \in \mathbb{N}$ and $\theta_0 < \Theta$. Split $a = a_1 + a_2$ at θ^-, and define $c_1 := 1 - a_1$. Write

$$d/c^n = d/(c_1 + a_2)^n = \frac{d/c_1^n}{(1 + a_2/c_1)^n} = \sum_i m_{i,n} \frac{da_2^i}{c_1^{i+n}},$$

for some natural numbers $m_{i,n}$. Note that $\operatorname{lt}(c_1) < \operatorname{lt}(c)$, and therefore, by induction on the length of c, each summand $x_i := m_{i,n} da_2^i / c_1^{i+n}$ splits at Γ. Moreover, for each $i \geq n_0$, $v(x_i) = v(m_{i,n}) + iv(a_2) \geq n_0\theta_0 > \Gamma$, and therefore x splits at Γ.

If $\hat{\Gamma} < \mathbb{Z}\Theta < \Gamma$, let $\Psi := \Gamma - \mathbb{Z}\Theta > 0$, and split $d = d_1 + d_2$ at Ψ. Note that $\mu(d_1) + \mathbb{Z}\mu(c^n) \leq \Psi + \mathbb{Z}\Theta \leq \Gamma$, and therefore $d_1/c^n \in B[\Gamma]$. It remains to split d_2/c^n. Let $\delta := v(d_2) > \Psi$. By definition of Ψ, there exists $n_0 \in \mathbb{N}$, $\theta < \Theta$, $\gamma > \Gamma$, such that $\delta \geq \gamma - n_0\theta$. Split $a = a_1 + a_2$ at θ^-, and define $c_1 := 1 - a_1$. As before,

$$d_2/c^n = d_2/(c_1 - a_2)^n = \frac{d/c_1^n}{(1 - a_2/c_1)^n} = \sum_i m_{i,n} \frac{da_2^i}{c_1^{i+n}},$$

and, by induction on the length of c, each summand $x_i := m_{i,n} d_2 a_2^i / c_1^{i+n}$ splits at Γ. Moreover, for every $i \geq n_0$,

$$v(x_i) = v(m_{i,n}) + v(d_2) + iv(a_2) \geq \delta + n_0\theta \geq \gamma > \Gamma,$$

and we are done.

In order to finish our proof, it now suffices to show that $B[\Gamma]\cap\mathcal{O}[\Gamma] = \{0\}$ for every cut Γ. In the next proposition, we will deduce a normal form for every non-zero element $b \in B[\Gamma]$ that will prove that $v(b) \le \Gamma$ so that b cannot lie in $\mathcal{O}[\Gamma]$. □

5.3 Proposition. *With the preceding definition of the tower $\mathcal{B}$, take $b \in B[\Gamma]$ for some $\Gamma \in \breve{G}$. Then b can be written in the form*

$$\sum_{i=1}^{k} \frac{r_i}{1+a_i}$$

with $a_i, r_i \in R$ such that $v(a_i) > 0$ and $\mu(r_i) + \mathbb{Z}\mu(a_i) \le \Gamma$ for all i, and such that

$$v(r_1) < \ldots < v(r_k) \quad \text{and} \quad \mathbb{Z}\mu(a_k) < \ldots < \mathbb{Z}\mu(a_1)\;.$$

Moreover, $v(r_{i+1}) > \mu(r_i) + \mathbb{Z}\mu(a_i)$ for $1 \le i < k$.

Proof. As a first step, we prove the following. Suppose that $a, a', r, r' \in R$ with $v(a) > 0$, $v(a') > 0$, $\mu(r) + \mathbb{Z}\mu(a) \le \Gamma$ and $\mu(r') + \mathbb{Z}\mu(a') \le \Gamma$. If

$$\max(\mu(r), \mu(r')) + \mathbb{Z}\mu(a) + \mathbb{Z}\mu(a') \le \Gamma\;, \tag{5.1}$$

then by parts 3.14–4 and 3.14–5 of Lemma 3.14 we have that

$$\begin{aligned}
&\mu(r(1+a') + r'(1+a)) + \mathbb{Z}\mu(a + a' + aa') \\
&\le \max(\mu(r) + \mu(1+a'), \mu(r') + \mu(1+a)) + \mathbb{Z}(\mu(a) + \mu(a')) \\
&= \max(\mu(r), \mu(r')) + \mathbb{Z}\mu(a) + \mathbb{Z}\mu(a') \le \Gamma
\end{aligned}$$

since $\mu(1+a) = \mu(a)$ and $\mu(1+a') = \mu(a')$. This implies that

$$\frac{r}{1+a} + \frac{r'}{1+a'} = \frac{r(1+a') + r'(1+a)}{1+a+a'+aa'} = \frac{r''}{1+a''}$$

with $a'', r'' \in R$, $v(a'') > 0$ and $\mu(r'') + \mathbb{Z}\mu(a'') \le \Gamma$.

If (5.1) does not hold, we must have that $\mathbb{Z}\mu(a) \ne \mathbb{Z}\mu(a')$, and if $\mathbb{Z}\mu(a)$ is the smaller of the two, we also must have that $\mu(r) + \mathbb{Z}\mu(a') > \Gamma$, which implies that $\mu(r) > \mu(r') + \mathbb{Z}\mu(a') =: \Theta$. In this case, split $r = s_1 + s_2$ at Θ so that $s_1 \in A[\Theta]$ and $s_2 \in \mathcal{O}[\Theta]$. It follows that $\mu(s_1) \le \mu(r') + \mathbb{Z}\mu(a')$ so that (5.1) holds with r replaced by s_1. So we can write

$$\frac{r}{1+a} + \frac{r'}{1+a'} = \frac{s_1(1+a') + r'(1+a)}{1+a+a'+aa'} + \frac{s_2}{1+a} = \frac{r''}{1+a''} + \frac{s_2}{1+a}$$

with $a'', a, r'', s_2 \in R$, $v(a'') > 0$, $v(a) > 0$, $\mu(r'') + \mathbb{Z}\mu(a'') \leq \Gamma$ and $\mu(s_2) + \mathbb{Z}\mu(a) \leq \Gamma$. We have that

$$\mu(r'') = \mu(s_1(1+a')+r'(1+a)) \leq \max(\mu(s_1)+\mu(1+a'), \mu(r')+\mu(1+a)) \leq \Theta,$$

hence if $r'' \neq 0$, then $v(s_2) > v(r'')$.

Let us also show that

$$\mathbb{Z}\mu(a) < \mathbb{Z}\mu(a'') \leq \mathbb{Z}\mu(a') \ .$$

Split $a' = a'_1 + a'_2$ at $\mathbb{Z}\mu(a)$ so that $a'_1 \in A[\mathbb{Z}\mu(a)]$ and $a'_2 \in \mathcal{O}[\mathbb{Z}\mu(a)]$; since $\mathbb{Z}\mu(a) < \mathbb{Z}\mu(a')$, we must have that $a'_2 \neq 0$. Then

$$1 + a'' = (1+a)(1+a') = (1+a)(1+a'_1+a'_2) = (1+a)(1+a'_1) + a'_2 + aa'_2.$$

Since $(1+a)(1+a'_1) \in A[\mathbb{Z}\mu(a)]$ and $0 \neq a'_2 + aa'_2 \in \mathcal{O}[\mathbb{Z}\mu(a)]$, we find that $\mathbb{Z}\mu(a) < \mu(a'')$ and hence $\mathbb{Z}\mu(a) < \mathbb{Z}\mu(a'')$. On the other hand, $\mathbb{Z}\mu(a'') = \mathbb{Z}\mu(a+a'+aa') \leq \mathbb{Z}\max(\mu(a), \mu(a'), \mu(a)+\mu(a')) = \mathbb{Z}(\mu(a)+\mu(a')) = \mathbb{Z}\mu(a')$.

Every non-zero element $b \in B[\Gamma]$ can be written as

$$b \ = \ \sum_{i=1}^{k} \frac{\tilde{r}_i}{1+\tilde{a}_i}$$

with $\tilde{a}_i, \tilde{r}_i \in R$ such that $v(\tilde{a}_i) > 0$ and $\mu(\tilde{r}_i) + \mathbb{Z}\mu(\tilde{a}_i) \leq \Gamma$ for all i. Suppose that this is a representation of b with minimal k. Then it follows that for any choice of i, j such that $1 \leq i < j \leq k$, (5.1) cannot hold for r_i, r_j, a_i, a_j in the place of r, r', a, a', respectively. So we know that all $\mathbb{Z}\mu(\tilde{a}_i)$ are distinct, and we may w.l.o.g. assume that $\mathbb{Z}\mu(\tilde{a}_k) < \ldots < \mathbb{Z}\mu(\tilde{a}_1)$.

Suppose that $k \geq 2$. Having that $\mathbb{Z}\mu(\tilde{a}_k) < \mathbb{Z}\mu(\tilde{a}_{k-1})$, we apply the above procedure to $\tilde{r}_k, \tilde{r}_{k-1}$, $\tilde{a}_k, \tilde{a}_{k-1}$ in the place of r, r', a, a', respectively. The elements r'' and s_2 obtained here cannot be zero since otherwise, k wouldn't have been minimal. We set $r_k = s_2$ and replace $\tilde{r}_{k-1}$ by r'' and $\tilde{a}_{k-1}$ by a''. By construction, $\mathbb{Z}\mu(a_k) < \mathbb{Z}\mu(\tilde{a}_{k-1})$ and $v(r_k) > \mu(\tilde{r}_{k-1}) + \mathbb{Z}\mu(\tilde{a}_{k-1})$. And we still have that $\mathbb{Z}\mu(\tilde{a}_{k-1}) < \mathbb{Z}\mu(\tilde{a}_{k-2})$. So now we repeat the above procedure with $\tilde{r}_{k-1}, \tilde{r}_{k-2}$, $\tilde{a}_{k-1}, \tilde{a}_{k-2}$ in the place of r, r', a, a', respectively. We note that the non-zero element s_2 we obtain this time, which will become our r_{k-1}, satisfies $\mu(s_2) = \mu(\tilde{r}_{k-1}) \leq \mu(\tilde{r}_{k-1}) + \mathbb{Z}\mu(\tilde{a}_{k-1}) < v(r_k)$. This yields $v(r_{k-1}) < v(r_k)$, and from now on we can proceed by descending induction. The element r'' found in the last step will then be our r_1. □

5.3 Extension to immediate field extensions

Let $\mathcal{A}$ be a (weak) tower of complements for $\mathbb{K}$, and $\mathbb{F}$ an *immediate* extension of $\mathbb{K}$. The aim of this subsection is to extend $\mathcal{A}$ to a (weak) tower of complements for $\mathbb{F}$, under some condition on the extension $\mathbb{F}/\mathbb{K}$. We need to study further properties of the map μ (cf. Definition 3.13).

5.4 Lemma. *Let $0 \neq a \in \mathbb{K}$ such that $va = 0$, $b := \frac{1}{a}$, $\Lambda := \mathbb{Z}\mu(a)$. Then, $\mu(b) \leq \Lambda$.*[6]

Proof. If $\Lambda = +\infty$, there is nothing to prove. Otherwise, decompose $b = b_1 + b_2$, with $b_1 \in A[\Lambda]$, $v(b_2) > \Lambda$. Hence, $\mu(ab_1) \leq \Lambda + \Lambda = \Lambda$, while $v(ab_2) = v(b_2) > \Lambda$. Moreover, $1 = ab = ab_1 + ab_2$, hence, by the uniqueness of the decomposition of 1 at Λ, we get $ab_2 = 0$, that is, $b = b_1 \in A[\Lambda]$. ☐

5.5 Lemma. *Let $a \in \mathbb{K}$ be of the form $a = a' + ct^\lambda$, where $c \in \mathbf{k}^\star$, and $0 = va' < \mu a' < \lambda$. Let $\Lambda := \mathbb{Z}\lambda^+$, $b := \frac{1}{a}$. Then, $\mu(b) = \Lambda$.*

Proof. Note that $\mu a = \lambda^+$. By the previous lemma, $\mu b \leq \Lambda$. Suppose, for a contradiction, that $\Gamma := \mu b < \Lambda$. Choose $n \in \mathbb{N}^\star$ such that $(n-1)\lambda > \Gamma$. Define

$$b' := \frac{1 - (1 - \frac{a}{a'})^n}{a} = \sum_{i=0}^{n-1} \binom{n}{i} a'^i (-a)^{n-1-i}.$$

Since $\mu a' < \lambda$ and $\mu a = \lambda^+$, $\mu b' \leq (n-1)\lambda^+$. Thus,

$$\mu(b - b') \leq \max\left\{ \mu b, \mu b' \right\} \leq \max\left\{ \Gamma, (n-1)\lambda^+ \right\} = (n-1)\lambda^+.$$

Therefore, $\mu\big(a(b - b')\big) \leq \mu a + \mu(b - b') \leq n\lambda^+$.

Moreover,

$$a(b - b') = \Big(\frac{a' - a}{a'}\Big)^n = \Big(\frac{-ct^\lambda}{a'}\Big)^n.$$

Hence, $v(b - b') = v\big(a(b - b')\big) = n\lambda$. Therefore, $b' = b + (b' - b)$, with $b \in A[\Gamma]$ and $b' - b \in \mathcal{O}[n\lambda^+] \subseteq \mathcal{O}[\Gamma]$. Thus, since $b' \in A[(n-1)\lambda^+]$, by 3.4–4, $b' - b \in A[(n-1)\lambda^+] \subseteq A[n\lambda^-]$. Finally, $b - b' \in \mathcal{O}[n\lambda^-] \cap A[n\lambda^-] = (0)$, so $b = b'$, that is, $a = a'$, a contradiction. ☐

5.6 Definition. For $a \in \mathbb{F}$, we define

$$\mathbf{\Lambda}_a := \left\{ v(a - c) : c \in \mathbb{K} \right\}^+ \in \breve{G}.$$

Note that if $a \in \mathbb{K}$ then $\mathbf{\Lambda}_a = +\infty$.

[6] That is, $b \in A[\Lambda]$.

5.7 Lemma. *Let* $\mathbf{R}$ *be a sub-ring of* $\mathbb{F}$ *such that* $\mathbb{K} \subseteq \mathbf{R} \subseteq \mathbb{F}$*, and* $a \in \mathbf{R} \setminus \mathbb{K}$*. Let* $\mathcal{B}$ *be a W.T.o.C. for* $\mathbf{R}$ *extending* $\mathcal{A}$*. Then,* $\boldsymbol{\Lambda}_a \leq \mu(a)$*.*

Proof. Suppose not. Let $\Gamma := \mu(a)$ and $c \in \mathbb{K}$ such that

$$\Gamma < v(a - c).$$

Decompose c at Γ: $c = c_1 + c_2$, with $c_1 \in A[\Gamma]$ and $c_2 \in \mathcal{O}[\Gamma]$. Since $a - c_1 = (a - c) + c_2$, we have

$$v(a - c_1) \geq \min\bigl(v(a - c), v(c_2)\bigr) > \Gamma.$$

Moreover, $a \in B[\Gamma]$ by definition of Γ, and $c_1 \in B[\Gamma]$, thus $a - c_1 \in B[\Gamma] \cap \mathcal{O}[\Gamma]$. Therefore, $a - c_1 = 0$, hence $a \in \mathbb{K}$, which is absurd. □

5.8 Corollary. *If* $\mathbb{F}$ *is contained in the completion of* $\mathbb{K}$*, then* $\mathcal{A}$ *itself is the unique W.T.o.C. which* $\mathbb{F}$ *extends* $\mathcal{A}$ *from* $\mathbb{K}$ *to* $\mathbb{F}$*. Moreover,* $\mathcal{B}$ *is multiplicative if and only if* $\mathcal{A}$ *is.*

Proof. **Existence**. Define $B[\Gamma] = A[\Gamma]$. All the properties of (weak) T.o.C. for $\mathcal{B}$ are obvious, except possibly the fact that $B[\Gamma] + \mathcal{O}[\Gamma] = \mathbb{F}$. Let $a \in \mathbb{F}$, and $c \in \mathbb{K}$ such that $v(a - c) > \Gamma$. Decompose $c = c_1 + c_2$, with $c_1 \in A[\Gamma]$ and $v(c_2) > \Gamma$. Then, $a = c_1 + (a - c + c_2)$. Therefore, $c_1 \in B[\Gamma]$ and $a - c + c_2 \in \mathcal{O}[\Gamma]$.

Uniqueness. If for some Γ we would have that there is some $a \in B[\Gamma] \setminus A[\Gamma]$ then we could decompose $a = a_1 + a_2$ with $a_1 \in A[\Gamma]$ and $a_2 \in \mathcal{O}[\Gamma]$. But then $0 \neq a - a_1 \in B[\Gamma] \cap \mathcal{O}[\Gamma]$, contradiction. □

We will consider the case when $\mathbb{F} := \mathbb{K}(a)$ for some $a \in \mathbb{F} \setminus \mathbb{K}$. Let $(a_\nu)_{\nu \in I}$ be a pseudo Cauchy sequence in $\mathbb{K}$, without a limit in $\mathbb{K}$, and converging to $a \in \mathbb{F}$. Note that in this case

$$\boldsymbol{\Lambda}_a = \bigl\{\, v(a - a_\nu) : \nu \in I \,\bigr\}^+.$$

We will say that a certain property of the sequence members a_ν holds *eventually* (or *for* ν *large enough*) if there is some $\nu_0 \in I$ such that it holds for all $\nu \geq \nu_0$, $\nu \in I$, and we will say that it holds *frequently* if for all $\nu' \in I$ it holds for some $\nu \in I$ with $\nu \geq \nu'$.

We will assume the reader to be familiar with the basic theory of pseudo Cauchy sequence as outlined in [Kap]. If $(a_\nu)_{\nu \in I}$ is of algebraic type, let m

be the degree of a minimal polynomial for it. Otherwise, let $m := +\infty$. In both cases, m is the maximum such that any polynomial $p(X) \in \mathbb{K}[X]$ of degree less than m will satisfy $v\big(p(a_\nu)\big) = v\big(p(a_\mu)\big)$ for ν, μ large enough.

5.9 Fundamental hypothesis. *a) Either $(a_\nu)_{\nu\in I}$ is of transcendental type (and therefore a is transcendental over $\mathbb{K}$),*

b) or a is a root of a minimal polynomial for $(a_\nu)_{\nu\in I}$.

In either case, $[\mathbb{F} : \mathbb{K}] = m$. Let $\mathbb{L} := \mathbb{K}[a]$, the ring generated by $\mathbb{K}$ and a. Every element of $\mathbb{L}$ can be written in a unique way as a polynomial in a of degree less than m. Moreover, if a is algebraic (case b)), then $\mathbb{L} = \mathbb{F}$. Decompose each $a_\nu = a'_\nu + a''_\nu$, where $a'_\nu \in A[\mathbf{\Lambda}_a]$, $a''_\nu \in \mathcal{O}[\mathbf{\Lambda}_a]$.

5.10 Remark. $\big(a'_\nu\big)_{\nu\in I}$ is a pseudo Cauchy sequence with the same limits as $(a_\nu)_{\nu\in I}$.

Hence, we can use $\big(a'_\nu\big)_{\nu\in I}$ instead of $(a_\nu)_{\nu\in I}$, and, w.l.o.g., we can assume that $a_\nu \in A[\mathbf{\Lambda}_a]$.

5.11 Definition. If $\mathbf{\Lambda}_a < +\infty$, for any $\Gamma \in \breve{G}$, define $B[\Gamma]$ to be the $\mathbf{k}$-linear subspace of $\mathbb{L}$ generated by $A[\Lambda]$, together with the monomials of the form

$$ca^n, \quad \text{where } n < m, \text{ and } c \in A[\Gamma - n\mathbf{\Lambda}_a]. \tag{5.2}$$

If instead $\mathbf{\Lambda}_a = +\infty$, define $B[\Gamma] := A[\Gamma]$.

5.12 Remark. Let $n < m$ and $c \in \mathbb{K}$. Then, $ca^n \in B[\Gamma]$ if and only if $c \in A[\Gamma - n\mathbf{\Lambda}_a]$.

Proof. The "if" direction follows from the definition of $\mathcal{B}$. Conversely, assume that $ca^n = c_1 a_1^{n_1} + \cdots + c_l a_l^{n_l}$, with $c_i \in A[\Gamma - n_i\mathbf{\Lambda}_a]$. Up to permuting and adding together some of the c_i's, we can assume that $n_1 < n_2 < \cdots < n_l < m$. Since the degree of a over $\mathbb{K}$ is n, if $c \neq 0$, then $\exists! k \leq l$ such that $n = n_k$; moreover, $c_i = 0$ for every $i \neq k$, and $c = c_{n_k}$. The conclusion follows. ☐

5.13 Lemma. *Assume that the family $\mathcal{A}$ is a tower of complements for $\mathbb{K}$. Let $b \in A[\Gamma]$, $c \in A[\Lambda - n\Gamma]$. Then, $cb^n \in A[\Lambda]$.*

Proof. Let $\Theta := (\Lambda - n\Gamma) + n\Gamma$. By Axiom **C**E, $cb^n \in A[\Theta]$. By Proposition 2.6, $\Theta \leq \Lambda$, therefore, by Axiom **C**D, $cb^n \in A[\Lambda]$. ☐

5.14 Proposition. *Assume that the family $\mathcal{A}$ is a tower of complements for $\mathbb{K}$. Then the family $\mathcal{B} := \{ B[\Gamma] : \Gamma \in \breve{G} \}$ defined above is a weak tower of complements for $\mathbb{L}$ extending $\mathcal{A}$.*

Proof. When $\mathbf{\Lambda}_a = +\infty$, our assertion follows from Corollary 5.8; therefore, we assume that $\mathbf{\Lambda}_a < +\infty$. Taking $n = 0$ in (5.2), we see that $A[\Gamma] \subseteq B[\Gamma]$, hence $\mathcal{B}$ extends $\mathcal{A}$.

Axiom **C**A is trivial, and Axiom **C**C is a consequence of the fact that $\mathcal{B}$ extends $\mathcal{A}$. Since $\Gamma - n\mathbf{\Lambda}_a$ is an increasing function of Γ, Axiom **C**D is also trivial.

Axiom **C**B splits into two parts: $B[\Gamma] + \mathcal{O}[\Gamma] = \mathbb{L}$ and $B[\Gamma] \cap \mathcal{O}[\Gamma] = (0)$. The first part is equivalent to that every polynomial $p(a) \in \mathbb{K}[a]$ of degree $n < m$ can be decomposed as $p(a) = p_1(a) + p_2(a)$, with $p_1(a) \in B[\Gamma]$, and $p_2(a) \in \mathcal{O}[\Lambda]$. We will prove this by induction on n. W.l.o.g., we can assume that $p(a)$ is a monomial ca^n.

If $\mathbf{\Lambda}_a < +\infty$, decompose $c = c_1 + c_2$, $c_1 \in A[\Gamma - n\mathbf{\Lambda}_a]$, $c_2 \in \mathcal{O}[\Gamma - n\mathbf{\Lambda}_a]$. By definition, $c_1 a^n \in B[\Gamma]$. Moreover, $v(c_2) \geq \gamma - n\lambda$, for some $\gamma > \Gamma$, $\lambda < \mathbf{\Lambda}_a$. Hence, $v(c_2) \geq \gamma - v\big((a - a_\nu)^n\big)$ for some $\nu \in I$. Therefore, $c_2(a - a_\nu)^n \in \mathcal{O}[\Gamma]$. Finally, we get

$$ca^n = \underbrace{c_1 a^n}_{\in B[\Gamma]} + \underbrace{c_2(a - a_\nu)^n}_{\in \mathcal{O}[\Gamma]} + c_2\big(a^n - (a - a_\nu)^n\big).$$

The polynomial (in a) $c_2\big(a^n - (a - a_\nu)^n\big)$ has degree less than n. Hence, by inductive hypothesis, it can be written as $b_1 + b_2$, with $b_1 \in B[\Gamma]$, $b_2 \in \mathcal{O}[\Gamma]$, and we get the decomposition of ca^n.

For the second part, assume that $q(a) := \sum_{n<m} c_n a^n \in \mathcal{O}[\Gamma] \cap B[\Gamma]$, that is, for every n, $c_n \in A[\Gamma - n\mathbf{\Lambda}_a]$, and $v\big(q(a)\big) > \Gamma$. Choose $\nu \in I$ large enough such that $v\big(q(a)\big) = v\big(q(a_\mu)\big)$. Since $a_\mu \in A[\mathbf{\Lambda}_a]$, $q(a_\mu) \in A[\Gamma]$, because, by the above lemma, each summand $c_n a_\mu^n$ is in $A[\Gamma]$. Hence, $q(a_\mu) = 0$ for every μ large enough. Therefore, $q(X)$ has infinitely many zeroes, so $q(X) = 0$, and in particular $q(a) = 0$.

Finally, to prove Axiom **C**F, observe that

$$\begin{aligned} ca^n \in B[\Gamma] &\iff c \in A[\Gamma - n\mathbf{\Lambda}_a] \iff t^\gamma c \in A[\gamma + \Gamma - n\mathbf{\Lambda}_a] \\ &\iff t^\gamma c a^n \in B[\gamma + \Gamma]. \end{aligned}$$

□

5.15 Lemma. *If $n < m$ and $c \in \mathbb{K}$, then $\mu(ca^n) = \mu(c) + n\mathbf{\Lambda}_a$. More generally, for every $c_0, \dots, c_{m-1} \in \mathbb{K}$,*

$$\mu(\sum_{0 \le n < m} c_n a^n) = \max_{n<m} \{ \mu(c_n) + n\mathbf{\Lambda}_a \}.$$

In particular, $\mu(a) = \mathbf{\Lambda}_a$.

Proof. By Lemma 3.14 and Lemma 2.4,

$$\begin{aligned}\mu(ca^n) &= \inf\{ \Lambda : ca^n \in B[\Lambda] \} = \inf\{ \Lambda : c \in A[\Lambda - n\mathbf{\Lambda}_a] \} = \\ &= \inf\{ \Lambda : \mu(c) \le \Lambda - n\mathbf{\Lambda}_a \} = \inf\{ \Lambda : \mu(c) + n\mathbf{\Lambda}_a \le \Lambda \} \\ &= \mu(c) + n\mathbf{\Lambda}_a.\end{aligned}$$

The second point is now obvious. □

For the rest of this section, we will assume that $\mathcal{A}$ is a T.o.C., and that $\mathcal{B}$ is built as in Proposition 5.14. unless explicitly said otherwise.

5.16 Lemma. *The family $\mathcal{B}$ is the unique W.T.o.C. on $\mathbb{L}$ such that:*

1. *$\mu(a) \le \mathbf{\Lambda}_a$;*
2. *for every $n < m$ and $c \in \mathbb{K}$, $\mu(ca^n) \le \mu(c) + n\mu(a)$.*

Proof. Let $\mathcal{B}'$ be another *W.T.o.C.* for $\mathbb{L}$ satisfying the conditions. By Lemma 5.7, $\mu(a) = \mathbf{\Lambda}_a$. Moreover, if $n < m$ and $c \in A[\Gamma - n\mathbf{\Lambda}_a]$,

$$ca^n \in B'[(\Gamma - n\mathbf{\Lambda}_a) + n\mathbf{\Lambda}_a] \subseteq B'[\Gamma].$$

Therefore, $B[\Gamma] \subseteq B'[\Gamma]$, and thus $\mathcal{B}' = \mathcal{B}$. □

5.17 Lemma. *Let $q(X) \in \mathbb{K}[X]$ such that $\deg q = n < m$. Then,*

1. *If $q(X) \neq 0$, then $v\big(q(a) - q(a_\nu)\big) > v(q(a)) = v(q(a_\nu))$ eventually.*
2. *$\mu(q(a_\nu)) \le \mu(q(a))$ for every $\nu \in I$.*
3. *$q(a) \in B[\Gamma]$ if and only if $q(a_\nu) \in A[\Gamma]$ eventually.*
4. *$\mu(q(a)) = \lim_{\nu \in I} \mu(q(a_\nu))$.*
5. *If $q(X) = q_1(X) + q_2(X)$, with $q_1(a) \in A[\Gamma]$ and $q_2(a) \in \mathcal{O}[\Gamma]$ such that $\deg q_i < m$, then $\deg q_i \le n$.*

Proof. For the first assertion, see [Kap].

By Lemma 5.15,

$$\mu(q(a)) = \max_{i \leq n} \{ \mu(b_i) + i\mathbf{\Lambda}_a \}.$$

Since $\mathcal{A}$ is multiplicative, $\mu(q(a_\nu)) \leq \max_i \{ \mu(b_i) + i\mu(a) \}$. Therefore, since $\mu(a_\nu) \leq \mathbf{\Lambda}_a$, we have $\mu(q(a_\nu)) \leq \mu(q(a))$.

Let $q(a_\nu) \in A[\Gamma]$ eventually. Decompose $q(a) = q_1(a) + q_2(a)$, with $q_1(a) \in B[\Gamma]$, and $q_2(a) \in \mathcal{O}[\Gamma]$. Then, $q_1(a_\nu)$ and $q(a_\nu)$ are in $A[\Gamma]$ eventually, and therefore $q_2(a_\nu) = 0$ eventually. Thus, $q_2(X)$ has infinitely many zeroes, hence $q_2 = 0$. Therefore, we have proved assertion 3.

For assertion 4, let $\Gamma' := \liminf_{\nu \in I} \mu(q(a_\nu))$. Then $\Gamma' \leq \mu(q(a))$. If, for a contradiction, $\Gamma' < \mu(q(a))$, choose $\Gamma > \Gamma'$ such that $\mu(q(a_\nu)) \leq \Gamma$ frequently. Decompose $q(a) = q_1(a) + q_2(a)$, with $q_1(a) \in A[\Gamma]$ and $0 \neq q_2(a) \in \mathcal{O}[\Gamma]$. Then by the third assertion, $q_1(a_\nu) \in A[\Gamma]$ and $q_2(a_\nu) \in \mathcal{O}[\Gamma]$ eventually, hence $q_2(a_\nu) = 0$ frequently, and therefore $q_2 = 0$, which is absurd.

Now consider $\Gamma'' := \limsup_{\nu \in I} \mu(q(a_\nu))$. Then $\Gamma'' \geq \liminf_{\nu \in I} \mu(q(a_\nu)) = \mu(q(a))$. By our second assertion, $\Gamma'' \leq \Gamma$; thus, $\lim_{\nu \in I} \mu(q(a_\nu))$ always exists and is equal to $\mu(q(a))$.

Assertion 5 is a consequence of the proof of Proposition 5.14. ☐

5.18 Lemma. *Let $\mathbf{\Lambda}_a < +\infty$, $n < m$, $d \in A[\mathbf{\Lambda}_a]$.*

1. *If $c \in A[\Lambda - n\mathbf{\Lambda}_a]$, then $c(a+d)^n \in B[\Lambda]$.*
2. *If $c \in A[\Lambda]$, then $c(a+d)^n \in B[\Lambda + n\mathbf{\Lambda}_a]$.*[7]

In particular, if $0 < n < m$, then $(a+d)^n \in B[n\mathbf{\Lambda}_a]$.

Proof. For the first part, write

$$c(a+d)^n = \sum_{i<n} \binom{i}{n} c d^{n-i} a^i. \tag{5.3}$$

Note that $\binom{i}{n} \in \mathbf{k}$. Hence, by Proposition 2.6,

$$\binom{i}{n} c d^{n-i} \in A[\Lambda - n\mathbf{\Lambda}_a] A[(n-i)\mathbf{\Lambda}_a] \subseteq A[(\Lambda - n\mathbf{\Lambda}_a) + (n-i)\mathbf{\Lambda}_a] \subseteq A[\Lambda - i\mathbf{\Lambda}_a].$$

[7] It is enough that $c \in A[(\Lambda + n\mathbf{\Lambda}_a) - n\mathbf{\Lambda}_a]$.

Therefore, by definition, each of the summands of (5.3) is in $B[\Lambda]$.

For the second part, define $\Lambda' := \Lambda + n\mathbf{\Lambda}_a$. By the first part, to prove the conclusion, it suffices to show that $c \in A[\Lambda' - n\mathbf{\Lambda}_a]$. By Proposition 2.6, $\Lambda' - n\mathbf{\Lambda}_a \geq \Lambda$, and the conclusion follows. $\square$

5.19 Corollary. *If $n < m$ and $c \in \mathbb{K}$, then, for every $\nu \in I$, $\mu\bigl(c(a-a_\nu)^n\bigr) = \mu(c) + n\mathbf{\Lambda}_a = \mu(ca^n)$.*

Proof. Since $a_\nu \in A[\mathbf{\Lambda}_a]$, we have that

$$c(a - a_\nu)^n \in B[\mu(c) + \ n\mathbf{\Lambda}_a]. \qquad \square$$

5.20 Proposition. *Let $p(X), q(X) \in \mathbb{K}[X]$. Assume that $\deg p + \deg q < m$, and that $p(a) \in B[\Gamma]$, $q(a) \in B[\Lambda]$. Then, $p(a)q(a) \in B[\Gamma + \Lambda]$.*

Proof. This is trivial if $\mathbf{\Lambda}_a = +\infty$. Otherwise, w.l.o.g. $p(a)$ and $q(a)$ are monomials of the form ca^n and da^r respectively, with $c \in A[\Gamma - n\mathbf{\Lambda}_a]$, $d \in A[\Lambda - r\mathbf{\Lambda}_a]$, and $n + r < m$. Then, $p(a)q(a) = cda^{n+r}$. Moreover, by Corollary 2.7,

$$cd \in A[\Gamma - n\mathbf{\Lambda}_a]A[\Lambda - r\mathbf{\Lambda}_a] \subseteq A[(\Gamma - n\mathbf{\Lambda}_a) + (\Lambda - r\mathbf{\Lambda}_a)] \subseteq A[(\Gamma + \Lambda) - (n+r)\mathbf{\Lambda}_a],$$

and the conclusion follows. $\square$

5.21 Corollary. *If, in the above proposition, $m = \infty$ (case a)), then $\mathcal{B}$ is a tower of complements for $\mathbb{L}$ (that is, it satisfies Axiom **CE**).*

Now our aim is to extend a tower of complements from $\mathbb{L} = \mathbb{K}[a]$ to $\mathbb{F} = \mathbb{K}(a)$. We could just use Proposition 5.2, but we we want to show how the construction works in the present special situation, where $\mathcal{A}$ is a tower of complements for $\mathbb{K}$, a an element of transcendental type over $\mathbb{K}$, satisfying case a) of the Fundamental Hypothesis (i.e., $m = \infty$). Let $\mathcal{B}$ be the W.T.o.C. defined in 5.11 (it is a T.o.C. by Prop. 5.14 and Corollary 5.21). Lemma 5.17 shows that an equivalent definition of $\mathcal{B}$ is given by:
for every $q(X) \in \mathbb{K}[X]$, $q(a) \in B[\Gamma]$ if and only if $q(a_\nu) \in A[\Gamma]$ eventually.

The above definition makes sense also in $\mathbb{F}$. Therefore, we define
$C[\Gamma] := \bigl\{\, r(a) : r(X) \in \mathbb{K}(X) \ \&\ r(a_\nu) \in A[\Gamma] \text{ eventually} \,\bigr\}$, and
$\mathcal{C} := \bigl(C[\Gamma]\bigr)_{\Gamma \in \check{G}}$.

5.22 Lemma. *The above defined family $\mathcal{C}$ is a T.o.C. for $\mathbb{F} = \mathbb{K}(a)$.*

Proof. All axioms are immediate to show, except **C**B and **C**E. The important fact used in the proof is that, for every $r(X) \in \mathbb{K}(X)$, $v\big(r(a)\big) = v\big(r(a_\nu)\big)$ eventually.

Let us prove Axiom **C**E first. Let $c \in C[\Gamma]$ and $c' \in C[\Gamma']$. Write $c = r(a)$ and $c' = r'(a)$, for some $r, r' \in \mathbb{K}(X)$. By definition, $r(a_\nu) \in A[\Gamma]$ and $r'(a_\nu) \in A[\Gamma']$ eventually. Thus, $rr'(a_\nu) \in A[\Gamma + \Gamma']$ eventually, and therefore $cc' \in C[\Gamma + \Gamma']$.

For Axiom **C**B, we prove first that $C[\Gamma] \cap \mathcal{O}[\Gamma] = \{0\}$. Let $c = r(a) \in C[\Gamma] \cap \mathcal{O}[\Gamma]$. Hence, $r(a_\nu) \in A[\Gamma]$ eventually. Moreover, $v\big(r(a)\big) = v\big(r(a_\nu)\big)$ eventually, and therefore $r(a_\nu) \in \mathcal{O}[\Gamma]$ eventually. Thus, $r(a_\nu) = 0$ eventually, hence $r(X)$ has infinitely many zeros, and so $r = 0$.

Now we prove that $\mathbb{F} = A[\Gamma] + \mathcal{O}[\Gamma]$ for every $\Gamma \in \breve{G}$. We have to show that every element $d/c \in \mathbb{F}$, where $d \in \mathbb{L}$ and $0 \neq c \in \mathbb{L}$, can be split at any given Γ. But if $\gamma = v(d/c)$, it suffices to show that $t^{-\gamma} d/c$ can be split at $-\gamma + \Gamma$. Hence we may assume w.l.o.g. that $v(d) = v(c) = 0$. We will prove, by induction on the length of c that, for every $0 < k \in \mathbb{N}$, and for every $\Gamma \in \breve{G}$, $b := d/c^k$ splits at Γ.

Let $d = p(a)$ and $c = q(a)$, for some $p, q \in \mathbb{K}[X]$. By part 2 of Lemma 5.17, we then have that $\mu(p(a_\nu)) \leq \mu(d)$ and $\mu(q(a_\nu)) \leq \mu(c)$ for all $\nu \in I$. By part 4 of the same lemma, $\mu(c) = \lim_{\nu \in I} \mu(q(a_\nu))$. Since $v(c) = 0$, $\mu(c)$ is a positive cut. Therefore, $\mu(q(a_\nu))$ is a positive cut $\leq \mu(c)$ eventually, showing that $\mathbb{Z}\mu(q^k(a_\nu)) \leq \mathbb{Z}\mu(c)$ eventually. Also, $v(p(a_\nu)) = v(d) = 0$ eventually.

Claim 1. If $\mu(d) + \mathbb{Z}\mu(c) \leq \Gamma$, then $d/c^k \in C[\Gamma]$.

Indeed, by the facts shown above, $\mu\big(p(a_\nu)\big) + \mu\big(q^k(a_\nu)\big) \leq \mu(d) + \mathbb{Z}\mu(c) \leq \Gamma$ eventually. Thus, by Lemma 5.4, $\mu\big((p/q^k)(a_\nu)\big) \leq \Gamma$ eventually. Therefore, $b \in C[\Gamma]$ by definition, and we are done.

Now split $d =: d_1 + d_2$ at Γ. Note that $v(d_2/c^k) = v(d_2) > \Gamma$, hence $d_2/c^k \in \mathcal{O}[\Gamma]$. Therefore, it suffices to prove that d_1/c^k splits at Γ; thus, we can assume that $d = d_1$, that is, $\mu(d) \leq \Gamma$. In view of this, Claim 1 constitutes the induction start for our induction on the length of c.

We write $c = 1 - \varepsilon$ for some $\varepsilon \in \mathbb{L}$ with $v(\varepsilon) > 0$. We assume that $\varepsilon \neq 0$ because otherwise, $b \in \mathbb{L}$, and there is nothing to show. Define $\Theta := \mu(c) = \mu(\varepsilon)$. Note that $\Theta > 0$ and $q(a_\nu) \in A[\Theta]$ eventually. There are 4 cases: $\mathbb{Z}\Theta \leq \widehat{\Gamma}$, or $\widehat{\Gamma} < \mathbb{Z}\Theta < \Gamma$, or $\mathbb{Z}\Theta = \Gamma$, or $\mathbb{Z}\Theta > \Gamma$.

If $\mathbb{Z}\Theta \leq \widehat{\Gamma}$, then, by Claim 1, $b \in C[\Gamma]$, and we are done.

If $\mathbb{Z}\Theta = \Gamma$, then $\mathbb{Z}\Theta = \widehat{\Gamma}$, and we are in the previous case.

If $\mathbb{Z}\Theta > \Gamma$ we have that $\Gamma < n_0\theta_0$ for some $\theta_0 < \Theta$ and $n_0 \in \mathbb{N}$. Split $\varepsilon =: \varepsilon_1 + \varepsilon_2$ at θ_0^-: i.e., $\mu(\varepsilon_1) < \theta_0 \leq v(\varepsilon_2)$. Define $c_1 := 1 - \varepsilon_1$. Note that, since $\theta_0 < \mu(c)$, $\mathrm{lt}(c_1) < \mathrm{lt}(c)$. Moreover,

$$b = \frac{d}{c^k} = \frac{d}{(c_1 - \varepsilon_2)^k} = \frac{d/c_1^k}{\big(1 - (\varepsilon_2/c_1)\big)^k} = \sum_{i \in \mathbb{N}} n_{i,k} \frac{d\varepsilon_2^i}{c_1^{i+k}},$$

for some natural numbers $n_{i,k}$. Since $\mathrm{lt}(c_1) < \mathrm{lt}(c)$, then, by induction on $\mathrm{lt}(c)$, each summand $n_{i,k} d\varepsilon_2^i / c_1^{i+k}$ splits at Γ. Moreover,

$$v\big(\sum_{i \geq n_0} n_{i,k} \frac{d\varepsilon_2^i}{c_1^{i+k}}\big) \geq v\big(\frac{d\varepsilon_2^{n_0}}{c_1^{n_0+k}}\big) = v(\varepsilon_2^{n_0}) \geq n_0\theta_0 > \Gamma,$$

and therefore b splits at Γ.

Finally, if $\widehat{\Gamma} < \mathbb{Z}\Theta < \Gamma$, let $\Psi := \Gamma - \mathbb{Z}\Theta > 0$, and split $d = d_1 + d_2$ at Ψ. Note that $\mu(d_1) + \mathbb{Z}\mu(c) \leq \Psi + \mathbb{Z}\Theta \leq \Gamma$, and therefore $d_1/c^k \in C[\Gamma]$. It remains to split d_2/c^k. Let $\delta := v(d_2) > \Psi$. By definition of Ψ, there exists $n_0 \in \mathbb{N}$, $\theta_0 < \Theta$, $\gamma > \Gamma$, such that $\delta \geq \gamma - n_0\theta_0$. Split $\varepsilon = \varepsilon_1 + \varepsilon_2$ at θ^-, and define $c_1 := 1 - \varepsilon_1$. As before,

$$d_2/c^k = d_2/(c_1 - \varepsilon_2)^n = \frac{d_2/c_1^k}{(1 - \varepsilon_2/c_1)^k} = \sum_i n_{i,k} \frac{d_2\varepsilon_2^i}{c_1^{i+k}}\ ,$$

and by induction on the length of c, each summand $n_{i,k} d_2 \varepsilon_2^i / c_1^{i+k}$ splits at Γ. Moreover, for every $i \geq n_0$,

$$v(n_{i,k} d_2 \varepsilon_2^i / c_1^{i+k}) = v(n_{i,k}) + v(d_2) + iv(\varepsilon_2) \geq \delta + n_0\theta \geq \gamma > \Gamma,$$

and we are done. □

5.23 Proposition. *Assume that $m < \infty$ (case b)). Let $p(X) := \sum_{n=0}^m b_n X^n \in \mathbb{K}[X]$ be the minimal polynomial of a over $\mathbb{K}$ (thus, $b_m = 1$). Then a necessary and sufficient condition for the family $\mathcal{B}$ to be multiplicative is that $\mathbf{\Lambda}_a = +\infty$ or $b_k \in A[m\mathbf{\Lambda}_a - k\mathbf{\Lambda}_a]$, for $k = 0, \dots, m-1$.*

Proof. **Necessity**. Since $a \in B[\mathbf{\Lambda}_a]$, if $\mathcal{B}$ is multiplicative, then $a^m \in B[m\mathbf{\Lambda}_a]$. Hence, $b_0 + b_1 a + \dots + b_{m-1} a^{m-1} \in B[m\mathbf{\Lambda}_a]$, and the conclusion follows from the definition of $B[m\mathbf{\Lambda}_a]$.

Sufficiency. This is trivial if $\mathbf{\Lambda}_a = +\infty$. Therefore, we can assume that $\mathbf{\Lambda}_a < +\infty$.

Take $q(X), q'(X) \in \mathbb{K}[X]$ of degrees $n, n' < m$ respectively, and cuts Λ, Λ' with $q(a) \in B[\Lambda]$ and $q'(a) \in B[\Lambda']$. We wish to show that $q(a)q'(a) \notin B[\Lambda + \Lambda']$. We write

$$q(X) := \sum_{i=0}^{n} c_i X^i,$$

$$q'(X) := \sum_{j=0}^{n'} c'_j X^j,$$

with $c_i \in A[\Lambda - i\mathbf{\Lambda}_a]$, $c'_j \in A[\Lambda' - j\mathbf{\Lambda}_a]$. Define $e := n + n' - m \in \mathbb{Z}$. We can assume that e is minimal. If $e < 0$, we have a contradiction with Proposition 5.20. Hence, $e \geq 0$. Then,

$$q(a)q'(a) = \sum_{i,j} c_i c'_j a^{i+j} = \underbrace{\sum_{i+j<m+e} c_i c'_j a^{i+j}}_{S_1} + \underbrace{a^{m+e} \sum_{i+j=m+e} c_i c'_j}_{S_2}.$$

It suffices to prove that each summand in the sum above is in $B[\Lambda + \Lambda']$. By definition of $\mathcal{B}$, $c_i a^i \in B[\Lambda]$ and $c'_j a^j \in B[\Lambda']$. Therefore, by minimality of e and Proposition 2.6, if $i + j < m + e$, then $c_i c'_j a^{i+j} \in B[\Lambda + \Lambda']$, hence $S_1 \in B[\Lambda + \Lambda']$.
Moreover, $a^{m+e} = -\sum_{k<m} b_k a^{k+e}$, hence

$$S_2 = -\sum_{k<m} b_k a^{k+e} \sum_{i+j=m+e} c_i c'_j. \tag{5.4}$$

It is enough to prove that $c_i c'_j b_k a^{k+e} \in B[\Lambda + \Lambda']$ for each $i + j = m + e$, $k < m$. Fix $l, l' < m$ such that $l + l' = k + e$. By Lemma 5.18,

$$a^l \in B[l\mathbf{\Lambda}_a].$$

By Lemma 2.9,

$$c_i c'_j b_k \in A[(\Lambda - i\mathbf{\Lambda}_a) + (\Lambda' - j\mathbf{\Lambda}_a) + (m\mathbf{\Lambda}_a - k\mathbf{\Lambda}_a)] \subseteq A[(\Lambda + \Lambda') - (k + e)\mathbf{\Lambda}_a]. \tag{5.5}$$

Hence, by Lemma 5.18 and Proposition 2.6,

$$c_i c'_j b_k a^{l'} \in B[((\Lambda + \Lambda') - (k + d)\mathbf{\Lambda}_a) + l'\mathbf{\Lambda}_a] \subseteq B[\Lambda + \Lambda' - l\mathbf{\Lambda}_a].$$

Since $l + l' < m + e$, we have that, by minimality of e and Proposition 2.6,

$$c_i c'_j b_k a^{k+e} = a^l \cdot c_i c'_j b_k a^{l'} \in B[((\Lambda + \Lambda') - l\mathbf{\Lambda}_a) + l\mathbf{\Lambda}_a] \subseteq B[\Lambda + \Lambda']. \quad \square$$

By Remark 2.10, in the case when $\mathbf{\Lambda}_a$ is of the form $\gamma + \widehat{\mathbf{\Lambda}_a}$ (in particular when it is the upper edge of a group), the hypothesis of Proposition 5.23 is equivalent to $b_k \in A[\mathbf{\Lambda}_a]$, for $k = 0, \dots, m-1$.

An important example when $\mathbf{\Lambda}_a$ is a group is when a is Henselian over $\mathbb{K}$ (that is, $va \geq 0$, and $p(X)$, the minimal polynomial of a, has coefficients in $\mathcal{O}$, and $v(\dot{p}(a)) = 0$), cf. [D].

5.24 Lemma. *Let $\mathcal{A}$ be a tower of complements for $\mathbb{K}$, and $\mathbb{K}^{\mathrm{H}}$ be the Henselization of $\mathbb{K}$. Then there is a tower of complements $\mathcal{B}$ for $\mathbb{K}^{\mathrm{H}}$ extending $\mathcal{A}$.*

Proof. Let $\mathbb{F}$ be a maximal subfield of $\mathbb{K}^{\mathrm{H}}$ such that:

- $\mathbb{K} \subseteq \mathbb{F}$;
- there exists a tower of complements for $\mathbb{F}$ extending $\mathcal{A}$.

If $\mathbb{K} = \mathbb{K}^{\mathrm{H}}$, we are done. Otherwise, we will reach a contradiction.

W.l.o.g., we can assume that $\mathbb{F} = \mathbb{K}$. Since $\mathbb{K}^{\mathrm{H}} \neq \mathbb{K}$, $\mathbb{K}$ does not satisfy Hensel's Lemma and hence there exists $c \in \mathbb{K}^{\mathrm{H}} \setminus \mathbb{K}$ such that c is Henselian over $\mathbb{K}$.

Let $p(X) \in \mathbb{K}[X]$ be the minimal polynomial of c over $\mathbb{K}$, and $\Lambda := \mathbf{\Lambda}_c = \sup\{ v(c-d) : d \in \mathbb{K} \}$. W.l.o.g., we can assume that the degree of p is minimal among the degrees of the minimal polynomials of elements Henselian over $\mathbb{K}$, and not in $\mathbb{K}$. Since $vc \geq 0$, $\Lambda > 0$. Decompose $p(X) = p_1(X) + p_2(X)$, with $p_1 \in A[\Lambda][X]$, $p_2 \in \mathcal{O}[\Lambda][X]$. Note that p_1 is monic and of the same degree as p, while $\deg p_2 < \deg p$. Moreover, $v(\dot{p}_1(c)) = 0$, because $v(\dot{p}(c)) = 0$, and $v(p_1 - p) > 0$.

Therefore, there exists $a \in \mathbb{K}^{\mathrm{H}}$ such that $p_1(a) = 0$, $va \geq 0$ and $v(c-a) > 0$. Thus, a is Henselian over $\mathbb{K}$.[8] Besides, by the minimality of the degree of p, either p_1 is irreducible, or $a \in \mathbb{K}$.

Claim 1. $a \in \mathbb{K}^{\mathrm{H}} \setminus \mathbb{K}$.

[8] Since the minimal polynomial of a is a divisor of p_1, and, by Gauß's lemma, it is in $\mathcal{O}[X]$.

In fact, by [F, Proposition 5.11][9], $v(a-c) \geq v(p-p_1) = v(p_2) > \mathbf{\Lambda}_c$. If a were in $\mathbb{K}$, this would contradict the definition of $\mathbf{\Lambda}_c$.

Therefore, p_1 is irreducible, and hence p_1 is the minimal polynomial of a over $\mathbb{K}$. Note moreover that $\mathbf{\Lambda}_c = \mathbf{\Lambda}_a$. Finally, by the observation preceding this lemma, we can extend $\mathcal{A}$ to a tower of complements for $\mathbb{K}(a)$, contradicting the maximality of $\mathbb{K}$. □

Concluding, let $\mathbb{F}$ be a Henselian valued field, with residue field $\mathbf{f}$ of characteristic 0, and value group G. There exists a residue field section $\iota : \mathbf{f} \to \mathbb{F}$, that we fix, and assume it is the inclusion map. There also exists a value group section $t : G \to \mathbb{F}^\star$, with factor set f. We claim that there exists a T.o.C. for $\mathbb{F}$ compatible with the choice of ι and of t.

5.25 Theorem. *Let $\mathbb{F}$ be a Henselian valued field, with residue field $\mathbf{f}$ of characteristic 0, and value group G. Assume that $\mathbb{F}$ contains its residue field $\mathbf{f}$ and let $t : G \to \mathbb{F}^\star$ be a section, with factor set f. Then there exists a T.o.C. for $\mathbb{F}$ compatible with the inclusion of $\mathbf{f}$ and with t.*

Proof. We have seen that we can build a T.o.C. for $\mathbf{f}(G, f)$. Let $\mathbb{K} \subseteq \mathbb{F}$ be a maximal subfield admitting a T.o.C. $\mathcal{A}$ and such that $\mathbf{f}(G, f) \subseteq \mathbb{K}$. Then by Lemma 5.24, $\mathbb{K}$ is Henselian. By Lemma 5.22, $\mathbb{F}$ is an algebraic extension of $\mathbb{K}$. Since this extension is immediate and $\operatorname{char} \mathbf{f} = 0$, this means that $\mathbb{F} = \mathbb{K}$. □

5.4 The case of positive residue characteristic

Towers of complements cannot always be extended to arbitrary immediate algebraic extensions.

5.26 Example. Let $\mathbf{f}$ be a perfect field of characteristic $p > 0$, and y, t be algebraically independent elements over $\mathbf{f}$. Let $\mathbf{k}$ be the perfect hull of $\mathbf{f}(y)$, and $\mathbb{K}$ be the perfect hull of $\mathbf{k}(t)$, with the t-adic valuation. $\mathbb{K}$ has residue field $\mathbf{k}$ and it is, in a canonical way, a truncation-closed subfield of $\mathbb{H} := \mathbf{k}((t^{\frac{\mathbb{Z}}{p^\infty}}))$.

[9]With $q := p_1$, $\alpha = 0$.

Let c, a, b be roots of the polynomials

$$\begin{aligned} &X^p - X - y, \\ &X^p - X - \frac{1}{t}, \\ &X^p - X - (\frac{1}{t} + y) \end{aligned}$$

respectively. As an element of $\mathbb{H}$, $a = t^{-\frac{1}{p}} + t^{-\frac{1}{p^2}} + t^{-\frac{1}{p^3}} + \cdots$. Moreover, $\mathbf{k}(c)$ is a proper algebraic extension of $\mathbf{k}$, and $b = a + c$.

It is easy to see that:

1. $\mathbb{K}(a)$ and $\mathbb{K}(b)$ are immediate algebraic extensions of $\mathbb{K}$.
2. $\mathbb{K}(a,b)$ is *not* an immediate extension of $\mathbb{K}$ (because $c \in \mathbb{K}(a,b)$, but it is not in $\mathbf{k}$).

Hence, $\mathbb{K}$ has (at least) 2 different maximal immediate algebraic extensions, one containing a, the other b. The truncation-closed embedding of $\mathbb{K}$ in $\mathbb{H}$ can be extended to a truncation-closed embedding of $\mathbb{K}(a)$, but not of $\mathbb{K}(b)$ (nor of any immediate extension of $\mathbb{K}(b)$).

However, both a and b satisfy the fundamental assumption b). The truncation-closed embedding of $\mathbb{K}$ in $\mathbb{H}$ induces a unique tower of complements $\mathcal{A}$. By Proposition 5.14, $\mathcal{A}$ extends to a W.T.o.C. for $\mathbb{K}(b)$, but no W.T.o.C. for $\mathbb{K}(b)$ extending $\mathcal{A}$ can be a T.o.C. Therefore, it is not possible to extend $\mathcal{A}$ to a tower of complements for $\mathbb{K}(b)$, but only to a *weak* tower of complements.

The element b does not satisfy the necessary and sufficient condition of Proposition 5.23, because $\mathbf{\Lambda}_a = \mathbf{\Lambda}_b = \sup_i\{\, -p^{-i} \,\} = 0^- < 0$, thus $\frac{1}{t} + y \notin A[p\Lambda_b] = A[0^-]$.

This example together with the condition given in Proposition 5.23 leads us the way to construct extensions of towers of complements to at least one suitable maximal immediate extension. For this, we need to determine "good" minimal polynomials associated with pseudo Cauchy sequences of algebraic type. From Theorem 13 of [Ku] we infer the following result, which is due to F. Pop:

5.27 Lemma. *Assume that $\mathbb{L}$ is a minimal immediate algebraic extension of the Henselian field $\mathbb{K}$ of characteristic $p > 0$, that is, it admits no proper* *subextension. Then* $\mathbb{L}$ *is generated by a root of an irreducible polynomial of the form*

$$p(X) \;=\; c + \sum_{i=0}^{n} c_i X^{p^i} \quad \textit{with} \;\; c_n = 1 \; .$$

Note that the polynomial

$$\mathcal{A}_p(X) \;:=\; p(X) - c \;=\; \sum_{i=0}^{n} c_i X^{p^i}$$

is additive, that is,

$$\mathcal{A}_p(x+y) \;=\; \mathcal{A}_p(x) + \mathcal{A}_p(y) \; .$$

We will now derive another minimal polynomial for an immediate extension of $\mathbb{K}$ from $p(X)$, one that is suitable for our purposes. Let a denote the root of $p(X)$ that generates $\mathbb{L}$, and choose a pseudo Cauchy sequence $(a_\nu)_{\nu\in I}$ without a limit in $\mathbb{K}$ that has a as a limit. We have

$$-p(a_\nu) \;=\; p(a) - p(a_\nu) \;=\; \sum_{i=1}^{n} c_i (a - a_\nu)^{p^i} \; .$$

Since the values $v(a - a_\nu)$ are eventually strictly increasing with ν, there is some i_0 (indepent of ν) such that for all $i \neq i_0$,

$$vc_{i_0}(a-a_\nu)^{p^{i_0}} \;=\; vc_{i_0} + p^{i_0} v(a-a_\nu) \;<\; vc_i + p^i v(a-a_\nu) \;=\; vc_i(a-a_\nu)^{p^i} \; .$$

We conclude that for large enough ν, the values

$$vp(a_\nu) \;=\; v\sum_{i=0}^{n} c_i (a-a_\nu)^{p^i} \;=\; vc_{i_0}(a-a_\nu)^{p^{i_0}}$$

are strictly increasing and are all contained in

$$vc_{i_0} + p^{i_0}\mathbf{\Lambda}_a \;\leq\; p^n \mathbf{\Lambda}_a \; ,$$

where the inequality follows from the case $i = n$ in the above inequality for the value of the summands. For $0 \leq i < n$, split

$$c_i \;=\; b_i + b'_i$$

at $p^n\mathbf{\Lambda}_a - p^i\mathbf{\Lambda}_a$ so that $b_i \in A[p^n\mathbf{\Lambda}_a - p^i\mathbf{\Lambda}_a]$ and $b_i' \in \mathcal{O}[p^n\mathbf{\Lambda}_a - p^i\mathbf{\Lambda}_a]$. As in the proof of Proposition 5.14 we find some $\nu_0 \in I$ such that for all i and all $\nu > \nu_0$,

$$vb_i' + p^i v(a_\nu - a_{\nu_0}) \;=\; vb_i' + p^i v(a - a_{\nu_0}) \;>\; p^n\mathbf{\Lambda}_a \;.$$

Now we split

$$c + \mathcal{A}_p(a_{\nu_0}) \;=\; b + b'$$

at $p^n\mathbf{\Lambda}_a$ so that $b \in A[p^n\mathbf{\Lambda}_a]$ and $b' \in \mathcal{O}[p^n\mathbf{\Lambda}_a]$. We set

$$q(X) \;:=\; b + \sum_{i=0}^{n} b_i X^{p^i} \in \mathbb{K}[X] \;.$$

Then for all $\nu > \nu_0$,

$$\begin{aligned}
q(a_\nu - a_{\nu_0}) \;&=\; b + \sum_{i=0}^{n} b_i (a_\nu - a_{\nu_0})^{p^i} \\
&=\; c + \sum_{i=0}^{n} c_i (a_\nu - a_{\nu_0})^{p^i} + \mathcal{A}_p(a_{\nu_0}) - b' - \sum_{i=0}^{n} b_i' (a_\nu - a_{\nu_0})^{p^i} \\
&=\; p(a_\nu) - b' - \sum_{i=0}^{n} b_i' (a_\nu - a_{\nu_0})^{p^i} \;.
\end{aligned}$$

Since $vp(a_\nu) < p^n\mathbf{\Lambda}_a$ and $v(b' + \sum_{i=0}^{n} b_i'(a_\nu - a_{\nu_0})^{p^i}) > p^n\mathbf{\Lambda}_a$, we conclude that

$$vq(a_\nu - a_{\nu_0}) \;=\; vp(a_\nu)$$

for all $\nu > \nu_0$. Hence, $vq(a_\nu - a_{\nu_0})$ is strictly increasing for large enough ν. On the other hand, $(a_\nu - a_{\nu_0})_{\nu\in I}$ is a pseudo Cauchy sequence without a limit in $\mathbb{K}$, like $(a_\nu)_{\nu\in I}$. Now we are ready to prove our main theorem for the case of positive characteristic:

5.28 Theorem. *Assume that the valued field $\mathbb{K}$ admits a tower of complements. Then there is at least one maximal immediate extension and at least one maximal immediate algebraic extension of $\mathbb{K}$ that admits a tower of complements that extends the tower of $\mathbb{K}$.*

Proof. Take any immediate extension $\mathbb{L}$ of $\mathbb{K}$ that admits an extension of the tower $\mathcal{A}$ of $\mathbb{K}$.

Claim: *If $\mathbb{L}$ is not algebraically maximal, then there is some proper immediate algebraic extension of $\mathbb{L}$ that admits an extension of the tower $\mathcal{A}$ and thus of the tower of $\mathbb{K}$.*

We may assume that $\mathbb{L}$ is Henselian because otherwise, we are done by Lemma 5.24. Take n to be minimal among all integers > 0 for which $\mathbb{L}$ admits an immediate extension of degree p^n. Take $\mathbb{L}'$ to be such an extension of degree p^n, and take $p(X)$ as in Lemma 5.27. Then choose the pseudo Cauchy sequence $(a_\nu)_{\nu\in I}$ and construct the polynomial $q(X)$ as described above. There cannot be any polynomial $r(X)$ of degree $< p^n$ such that $vr(a_\nu - a_{\nu_0})$ eventually increases with ν since otherwise by Theorem 3 of [Kap], there would be an immediate extension of degree $< p^n$ of $\mathbb{L}$ (note that any immediate algebraic extension of $\mathbb{L}$ has degree a power of p). Hence again by Theorem 3 of [Kap], we may choose any root $\tilde{a}$ of $q(X)$ and an immediate extension of v from $\mathbb{L}$ to $\mathbb{L}(\tilde{a})$. Since the coefficients of $q(X)$ satisfy the conditions of Proposition 5.23, the tower $\mathcal{A}$ extends to a tower of complements of $\mathbb{L}(\tilde{a})$. This proves our claim and, by means of Zorn's Lemma, the algebraic part of our theorem.

If $\mathbb{L}$ is not maximal, then by what we have shown, it admits an extension of the tower to some maximal immediate algebraic extension. If it is already algebraically maximal but admits a proper immediate extension generated by a limit a of a transcendental pseudo Cauchy sequence, then Lemma 5.22 shows that the tower of $\mathbb{L}$ can be extended to a tower of $\mathbb{L}(a)$. Again by means of Zorn's Lemma, this proves the remaining part of our theorem. □

Since the maximal immediate algebraic extensions of Kaplansky fields are unique up to (valuation preserving) isomorphism, we obtain:

5.29 Theorem. *Let $\mathbb{F}$ be an algebraically maximal Kaplansky field. Assume that $\mathbb{F}$ contains its residue field $\mathbf{f}$ and let $t : G \to \mathbb{F}^\star$ be a section, with factor set f. Then there exists a T.o.C. for $\mathbb{F}$ compatible with the inclusion of $\mathbf{f}$ and with t.*

Since algebraically maximal Kaplansky fields of positive characteristic, being perfect fields, always admit embeddings of their residue field and a value group section, we conclude:

5.30 Corollary. *Every algebraically maximal Kaplansky field of positive characteristic with value group G and residue field $\mathbf{k}$ admits a truncation closed embedding in some power series field $\mathbf{k}((G, f))$. In particular, every* *algebraically closed valued field of positive characteristic with value group G and residue field $\mathbf{k}$ admits a truncation closed embedding in $\mathbf{k}((G))$.*

References

[B-K-K] Biljakovic, Darko - Kotchetov, Mikhail - Kuhlmann, Salma: *Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields*, in: Logic, Algebra and Arithmetic, Lecture Notes in Logic, **Vol 26**, Association for Symbolic Logic, AK Peters (2006), 42–65

[D] Delon, Francoise: *Quelques propriétés des corps valués en théories des modèles*, Thèse d'état. Paris 7, (1982)

[E] Endler, Otto: *Valuation theory*, Springer, Berlin (1972)

[E-P] Engler, Antonio J. and Prestel, Alexander: *Valued Fields*, Monographs in Mathematics, Springer (2005)

[F] Fornasiero, Antongiulio: *Embedding Henselian fields into power series*, Journal of Algebra **304** (2006), 112-156

[F-M] Fornasiero, Antongiulio and Mamino, Marcello: *Arithmetic of Dedekind cuts of ordered Abelian groups*, Annals of Pure and Applied Logic, **156** (2008), 210–244

[Kap] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. J. **9** (1942), 303–321

[Ku] Kuhlmann, F.-V.: *Additive Polynomials and Their Role in the Model Theory of Valued Fields*, in: Logic, Algebra and Arithmetic, Lecture Notes in Logic, **Vol 26**, Association for Symbolic Logic, AK Peters (2006), 160–203

[Ku2] Kuhlmann, F.-V.: *Dense subfields of Henselian fields, and Integer parts*, in: Logic, Algebra and Arithmetic, Lecture Notes in Logic, **Vol 26**, Association for Symbolic Logic, AK Peters (2006), 204–226

[M-R] Mourgues, Marie-Hélène and Ressayre, Jean-Pierre: *Every real closed field has an integer part*, J. Symbolic Logic **58** (1993), 641–647

[Ri] Ribenboim, Paulo: *Théorie des valuations*, Les Presses de l'Université de Montréal, Montréal (1968)

[S] Shepherdson, J. C.: *A Non-Standard Model for a Free Variable Fragment of Number Theory*, Bulletin De L'Academie Polonaise Des Sciences, vol. **XII**, (1964)

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