REAL ALGEBRAIC GEOMETRY LECTURE NOTES (05: 23/04/15)

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Contents

1. Valuation basis

Definition 1.1. $\mathcal{B} \subseteq V \backslash \{0\}$ is a **Q-valuation basis** of (V, v) if

(1) $\mathcal{B}$ is a Q-linear basis for V,
(2) $\mathcal{B}$ is Q-valuation independent.

Remark 1.2. $\mathcal{B}$ is a Q-valuation basis $\Rightarrow$ $\mathcal{B}$ is maximal valuation independent.

(This is because valuation independence $\Rightarrow$ linear independence).

Warning 1.3.

(i) a maximal valuation independent set needs not to be a valuation basis.
Example: $\mathrm{H}_{\mathbb{N}}\, \mathbb{Q}$ is a $\mathbb{Q}$-vector space, with $v_{\min}$ valuation. Consider

$$\mathcal{B} = \{(1, 0, \ldots), (0, 1, \ldots), \ldots\} \subseteq \mathrm{H}_{\mathbb{N}}\, \mathbb{Q} \backslash \{0\}.$$

Then $\forall \gamma \in \mathbb{N} : \mathcal{B}_\gamma = \{1\}$, which is a $\mathbb{Q}$-basis of $B(\gamma)$. Hence, $\mathcal{B}$ is maximal valuation independent. However, note that $\mathcal{B}$ is not a $\mathbb{Q}$-linear basis of $\mathrm{H}_{\mathbb{N}}\, \mathbb{Q}$.

(ii) a valued vector space needs not to admit a valuation basis.

Example 1.4. $(\coprod_{\gamma \in \Gamma} B(\gamma),\ v_{\min})$ admits a valuation basis.

Proof. Let $\mathcal{B}_\gamma$ be a Q-basis of $B(\gamma)$ for all $\gamma \in \Gamma$ and consider

$$\mathcal{B} := \bigcup_{\gamma \in \Gamma} \{b\chi_{\{\gamma\}};\ b \in \mathcal{B}_\gamma\},$$

where $\forall\, \gamma \in \Gamma$

$$\chi_\gamma \colon \Gamma \longrightarrow Q$$

$$\chi_\gamma(\gamma') = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma'. \end{cases}$$

□

Corollary 1.5. *Let (V, v) be a valued $\mathbb{Q}$-vector space with skeleton $S(V) = [\,\Gamma, \{B(\gamma) : \gamma \in \Gamma\}\,]$. Then (V, v) admits a valuation basis if and only if*

$$(V, v) \;\cong\; (\coprod_{\gamma\in\Gamma} B(\gamma),\; v_{\min}).$$

Proof.

($\Leftarrow$) ÜA.

($\Rightarrow$) Let $\mathcal{B} := \{b_i : i \in I\}$ be a valuation basis for (V, v). Then $\mathcal{B}$ is maximal valuation independent. For every $b_i \in \mathcal{B}$ with $v(b_i) = \gamma$ define

$$h(b_i) = \pi(\gamma, b_i)\chi_\gamma \in \coprod_{\gamma\in\Gamma} B(\gamma)$$

and then extend h to all of V by linearity, i.e. for $x \in V$ such that $x = \sum_{b_i\in\mathcal{B}} q_{b_i} b_i$ define

$$h(x) := \sum_{b_i\in\mathcal{B}} q_{b_i} h(b_i).$$

Verify that h is valuation preserving, i.e. verify that

$$v_{\min}(h(x)) = v(x) \;(= \mathrm{id}(v(x))) \quad \forall x \in V$$

First consider the case $x = b_i$. Then it holds by construction $v(b_i) = v_{\min}(h(b_i))$.

For arbitrary x we have $h(x) = \sum q_{b_i} h(b_i)$, and therefore

$$\begin{aligned} v(x) &= \min\{v(b_i) : b_i \in \mathcal{B}\} \\ &= \min\{v_{\min}(h(b_i)) : b_i \in \mathcal{B}\} \\ &= v_{\min}(h(x)). \end{aligned}$$

□

Corollary 1.6. *Let (V, v) be a valued $\mathbb{Q}$-vector space with skeleton $S(V) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. Then*

$$(\coprod_{\gamma\in\Gamma} B(\gamma),\; v_{\min}) \;\hookrightarrow\; (V, v),$$

i.e. there exists a valued subspace (V_0, v_0) of (V, v) such that

$$(V_0, v_0) \cong (\coprod B(\gamma, v_{\min})).$$

Proof. By Zorn's lemma, let $\mathcal{B} \subset V \setminus \{0\}$ be maximal valuation independent. Set

$$V_0 := \langle\, \mathcal{B} \,\rangle_Q.$$

Then $\mathcal{B}$ is a valuation basis of V_0 and the extension $V_0 \subseteq V$ is immediate by maximality. By definition $S(V_0) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. So $(V_0, v_{|V_0})$ admits a valuation basis and has skeleton $S(V_0) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. By the previous corollary $(V_0, v_{|V_0}) \cong (\coprod B(\gamma), v_{\min})$. □