Stability of Minimal Supersolutions of Convex BSDEs

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June 28, 2013

We study the stability of the minimal supersolution of a convex backward stochastic differential equation with respect to the generator. More precisely, we give conditions under which the nonlinear operator of mapping the generator to the minimal supersolution is lower semicontinuous. To that end we prove results on the stability of closed convex hulls of a sequence of convex functions.
Keywords: Stability of Minimal Supersolutions of Convex BSDEs, Stability of Closed Convex Hulls

1 Introduction

On a filtered probability space, where the filtration is generated by a d-dimensional Brownian motion W, a supersolution of a *backward stochastic differential equation (BSDE)* is given by a càdlàg *value process* Y and a *control processes* Z, such that

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi,$$

for all $0 \leq s \leq t \leq T$. Here the *terminal condition* ξ is a random variable, and the *generator* g a measurable function of (y, z). The main objective of this work is to extend results on the stability of the minimal supersolution with respect to pertubations of the generator obtained recently in Drapeau et al. [4]. More precisely, we derive stability theorems for sequences of generators which are not necessarily increasing as assumed in [4, Theorem 4.14].

To that end, we first establish a stability concept that is associated with an operation of convex analysis, known as *closed convex hull operation*: Given a sequence $(g^k : \mathbb{R}^n \to \overline{\mathbb{R}})$ of functions, we define the *related sequence* $(\overline{\text{conv}}(g^m)_{m\geq k} : \mathbb{R}^n \to \overline{\mathbb{R}})$ *of closed convex hulls*, where, for all $k \in \mathbb{N}$, $\overline{\text{conv}}(g^m)_{m\geq k}$ is the greatest convex and lower semicontinuous function that is majorized by g^m, for all $m \geq k$. Assuming

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that (g^k) converges pointwise to a proper function g, we give a sufficient and a necessary condition for the pointwise convergence of $(\overline{\operatorname{conv}}(g^m)_{m\geq k})$ to g. Our main idea is to employ a notion of set convergence for the related epigraphs which is known as *epi-convergence* in the sense of *Painlevé-Kuratowski*. In addition, a convenient representation for $(\overline{\operatorname{conv}}(g^m)_{m\geq k})$ allows to combine results of the theories of convex duality and epi-convergence. We are not aware of existing literature for this specific problem, but we benefit from numerous statements concerning convex, lower semicontinuous functions in connection with concepts from 'Variational Analysis' by R.T. Rockafellar and R. J.-B. Wets.

In the second part of this paper we use these results to investigate the stability of the minimal supersolution with respect to the generator. The setting we work in is inspired by [4], and their Theorem 4.14 is our starting point. It roughly states that for a monotone increasing sequence of generators (g^k) converging to a generator g the corresponding sequence of minimal supersolutions increases and converges to the minimal supersolution associated with g. The objective of the present paper is to complement these results by dropping the assumption that the sequence (g^k) is monotone increasing. We prove that in this case one still obtains lower semicontinuity of the operator that maps the generator to the minimal supersolution. The main idea is to find an increasing sequence of generators (h^k) converging to g such that $h^k \leq g^m$, for all $m \geq k$, and then to apply [4, Theorem 4.14]. Since we work with generators that are convex in the control variable it is natural to define h^k as the convex hull of $(g^m)_{m\geq k}$. Under some assumptions, which also rely on the results found in the first part of the paper, the sequence (h^k) has the desired properties.

This work is organized as follows. We collect some notation in Section 2. Then, in Section 3, we give results on the stability of closed convex hulls. Finally, we prove our main results on stability of the minimal supersolution in Section 4.

2 Preliminaries and notations

Given a set $C \subset \mathbb{R}^n$, its *convex hull* is the smallest convex set including C, and we denote it by $\operatorname{conv}(C)$. Accordingly, the *closed hull*, $\operatorname{cl}(C)$, of C, is defined as the smallest closed superset of C, and $\overline{\operatorname{conv}}(C) := \operatorname{cl}(\operatorname{conv}(C))$ is the smallest closed and convex set that contains C, the so-called *closed convex hull of* C, see [3, Theorem 3.6.]. Given an extended real-valued function $g : \mathbb{R}^n \to \bar{\mathbb{R}}$, we denote its *epigraph* by $\operatorname{epi}(g) := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq g(x)\}$, and we call g *proper* if its *domain* $\operatorname{dom}(g) := \{x \in \mathbb{R}^n : g(x) < \infty\}$ is non-empty and $g(x) > -\infty$, for all $x \in \mathbb{R}^n$. Furthermore, we say that g is *lower semicontinuous*, or simply *closed*, if its epigraph $\operatorname{epi}(g)$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}$. The *lower closure* or *closed hull* of a function g, denoted by $\operatorname{cl}(g)$, is defined to be the greatest lower semicontinuous function that is majorized by g. An equivalent characterization is given by the set equality $\operatorname{epi}(\operatorname{cl}(g)) = \operatorname{cl}(\operatorname{epi}(g))$, since, by definition, the set $\operatorname{cl}(\operatorname{epi}(g))$ is the smallest closed and epigraphical set that includes $\operatorname{epi}(g)$. Moreover, a function g is said to be *convex* if its epigraph is a convex subset of $\mathbb{R}^n \times \mathbb{R}$. Given a function g, its *convex hull*, $\operatorname{conv}(g)$, is defined as the greatest convex function that is majorized by g. It is straightforward to verify that $\operatorname{epi}(\operatorname{conv}(g))$ is equivalently characterized as the smallest convex and epigraphical set that includes $\operatorname{epi}(g)$, see [9, p.33+]. Furthermore, given a sequence (g^k) of functions, the two hull concepts of lower semicontinuity and convexity can be combined to define the sequence $(\overline{\operatorname{conv}}(g^m)_{m\geq k})$ of *closed convex hulls*, where, for all $k \in \mathbb{N}$, $\overline{\operatorname{conv}}(g^m)_{m\geq k}$ is the greatest convex, lower semicontinuous function that fulfills

$$\overline{\operatorname{conv}}(g^m)_{m\geq k}(x) \leq g^l(x), \quad \text{for all } x \in \mathbb{R}^n, \text{ and all } l \geq k. \tag{2.1}$$

In this context, a convenient representation is given by

$$\overline{\operatorname{conv}}(g^m)_{m\geq k}(x) = (\inf_{m\geq k} g^m)^{**}(x), \text{ for all } x \in \mathbb{R}^n, \text{ all } k \in \mathbb{N}, \tag{2.2}$$

if either $\overline{\text{conv}}(g^m)_{m\geq 1}(x) > -\infty$, for all $x \in \mathbb{R}^n$, or the convex biconjugate of $(\inf_{m\geq 1} g^m)$ satisfies $(\inf_{m\geq 1} g^m)^{**}(x) > -\infty$, for all $x \in \mathbb{R}^n$, see [6, Theorem 4.1.2]. Moreover, we recall that the convex conjugation or *Legendre-Fenchel transformation* is an involution on the space of proper, convex, lower semicontinuous functions. Hence, given a proper, convex, lower semicontinuous function g, its *biconjugate* $g^{**} : \mathbb{R}^n \to \overline{\mathbb{R}}$ fulfills

$$g(x) = g^{**}(x) := \sup_{y\in\mathbb{R}^n} \{\langle y, x\rangle - g^*(y)\}, \quad \text{for all } x \in \mathbb{R}^n, \tag{2.3}$$

where $g^*(y) := \sup_{x\in\mathbb{R}^n}\{\langle y, x\rangle - g(x)\}$, for all $y \in \mathbb{R}^n$, is also proper, convex, and lower semicontinuous, see for example [10, Theorem 11.1].

Throughout this work, if extended arithmetic is applied, we regard $\infty + \rho$ as ∞ for any real ρ, and we use the so-called *inf-addition*, that is, $\infty+(-\infty) = (-\infty)+\infty = \infty$. Furthermore, we need the notion of *pointwise convergence* to be applicable to extended real-valued functions: A sequence (g^k) of functions is said to converge pointwise to a function g at $x \in \mathbb{R}^n$ if $\limsup_k g^k(x) = \liminf_k g^k(x) = g(x)$, and to converge pointwise on a set D if this is true for every $x \in D$. The case $D = \mathbb{R}^n$ is denoted by $g^k \xrightarrow{p} g$ or $\lim_k g^k = g$. Moreover, we understand inequalities and strict inequalities between any two functions g^1, g^2 in the pointwise sense, that is, $g^1 \leq (<) g^2$ is equivalent to $g^1(x) \leq (<) g^2(x)$, for all $x \in \mathbb{R}^n$.

In addition, we employ the notion of *epi-convergence* in the sense of Painlevé-Kuratowski: A sequence (g^k) of functions is *epi-convergent* to a function g, denoted by $g = e\text{-}\lim_k g^k$ or $g^k \xrightarrow{e} g$, if at each point $x \in \mathbb{R}^n$ holds true that

$$\begin{cases} \text{there exists a sequence } (x^k) \subset \mathbb{R}^n, \text{ with } x^k \xrightarrow[k\to\infty]{} x, \text{ such that } g(x) \geq \limsup_{k\to\infty} g^k(x^k), \\ \text{for all sequences } (x^k) \subset \mathbb{R}^n, \text{ with } x^k \xrightarrow[k\to\infty]{} x, \text{ holds } g(x) \leq \liminf_{k\to\infty} g^k(x^k). \end{cases} \tag{2.4}$$

Without any further condition, epi-convergence neither implies nor is implied by pointwise convergence, see [1, p.156+], but in the case of an increasing sequence (g^k) of lower semicontinuous functions it is known that

$$e\text{-}\lim_k g^k = \sup_k(\text{cl}(g^k)) = \sup_k g^k = \lim_k g^k, \tag{2.5}$$

see for example [10, Proposition 7.4]. In addition, the following relation between epi-convergence and pointwise convergence will be used in the next sections:

Theorem 2.1. [10, Theorem 7.17, Theorem 11.34] *Let g and (g^k), $k \in \mathbb{N}$, be proper, convex, lower semicontinuous functions. If $\text{int}\,\text{dom}(g) \neq \emptyset$, then the following statements are equivalent:*

(a) $g = e\text{-}\lim g^k$,

(b) $g^* = e\text{-}\lim(g^k)^*$,

(c) there is a dense subset D of $\mathbb{R}^n$ such that $g^k(x) \to g(x)$, for all $x \in D$.

3 Stability of closed convex hulls

In this section we study properties of a stability concept that is associated with the closed convex hull operation: Given a sequence (g^k) of functions which converges pointwise to a proper function g, we state conditions for the pointwise convergence of $(\overline{\text{conv}}(g^m)_{m\geq k})$ to g. First, we prove a sufficient condition that will provide stability statements for minimal supersolutions of convex BSDEs in the next section. In addition, we give a necessary condition for the pointwise convergence of $(\overline{\text{conv}}(g^m)_{m\geq k})$ to g.

Theorem 3.1. *Let (g^k) be a sequence of convex, lower semicontinuous functions converging pointwise to a lower semicontinuous function g. If $\operatorname{int}\operatorname{dom}(g) \neq \emptyset$ and $\operatorname{int}\operatorname{dom}(g^*) \neq \emptyset$, then $(\overline{\operatorname{conv}}(g^m)_{m\geq k})$ converges pointwise to g.*

Proof. First, we deduce that g is convex, since it is the pointwise limit of convex functions, see [10, Proposition 2.9]. Moreover, if $\operatorname{int}\operatorname{dom}(g) \neq \emptyset$ and $\operatorname{int}\operatorname{dom}(g^*) \neq \emptyset$, then g is proper: Suppose the opposite. Then, along with the assumption that $\operatorname{int}\operatorname{dom}(g) \neq \emptyset$, there exists $x \in \mathbb{R}^n$ such that $g(x) = -\infty$. It follows that $g^* \equiv +\infty$, in contradiction to $\operatorname{int}\operatorname{dom}(g^*) \neq \emptyset$. Hence, g is proper. As a consequence, given $x \in \operatorname{dom}(g)$, the convex, lower semicontinuous function g^k is finite at x and thus proper, for all $k \geq \tilde{k}$ for some suitable $\tilde{k} \in \mathbb{N}$, see for instance [2, Proposition 2.111]. A consecutive application of Theorem 2.1 yields $(g^k)^* \xrightarrow{e} g^*$, providing a dense subset D of $\mathbb{R}^n$ such that $(g^k)^*(x) \to g^*(x)$, for all $x \in D$. Now, it is straightforward to verify that we may deduce

$$(\inf_{m\geq k} g^m)^*(x) = \sup_{m\geq k} (g^m)^*(x) \to g^*(x), \quad \text{for all } x \in D, \tag{3.1}$$

see for instance [10, Theorem 11.23]. The conjugate $(\inf_{m\geq k} g^m)^*$ is even proper, for all $k \geq \hat{k}$ for some suitable $\hat{k} \in \mathbb{N}$: Given $x \in \operatorname{int}\operatorname{dom}(g^*)$, there exists a real $\epsilon > 0$ with $B_\epsilon(x) \subseteq \operatorname{dom}(g^*)$ and $B_\epsilon(x) \cap D \neq \emptyset$, according to the density property of $D \subseteq \mathbb{R}^n$. Take an arbitrary element $y \in B_\epsilon(x) \cap D$. Hence, $g^*(y) \in \mathbb{R}$ by standard arguments, and there exists $\hat{k} \in \mathbb{N}$ such that $(\inf_{m\geq k} g^m)^*(y)$ is finite, for all $k \geq \hat{k}$. Here again, the result follows from [2, Proposition 2.111]. As a consequence, relation (3.1) along with an application of Theorem 2.1 yield $(\inf_{m\geq k} g^m)^{**} \xrightarrow{e} g^{**}$, where $g^{**} = g$ by the involution property (2.3). In addition, it follows from (2.5) that $(\inf_{m\geq k} g^m)^{**} \xrightarrow{p} g$ such that, by means of the representation (2.2), for all $k \geq \hat{k}$,

$$\overline{\operatorname{conv}}(g^m)_{m\geq k} = (\inf_{m\geq k} g^m)^{**} \xrightarrow{p} g.$$

□

Theorem 3.1 will be essential for the proof of Theorem 4.2 which gives a stability result in a BSDE context. We complete this section by showing a necessary condition for the pointwise convergence of $(\overline{\operatorname{conv}}(g^m)_{m\geq k})$ to g.

Proposition 3.2. *Let (g^k) be a sequence of functions. If $(\overline{\operatorname{conv}}(g^m)_{m\geq k})$ converges pointwise to a proper function g, then $\operatorname{cl}(\inf_k(\inf_{m\geq k} g^m)^*) = g^*$. If, in addition, the function $\inf_k(\inf_{m\geq k} g^m)^*$ is lower semicontinuous at 0, then*

$$\lim_{k\to\infty} \inf_{m\geq k} \inf_{\mathbb{R}^n} g^m = \inf_{\mathbb{R}^n} g.$$

Proof. The function g is clearly convex and lower semicontinuous as the pointwise limit of an increasing sequence of convex, lower semicontinuous functions, see [10, Proposition 2.9] and [10, Proposition 1.26]. Moreover, given $x \in \operatorname{dom}(g)$, it follows that, for all $k \geq \tilde{k}$ for some suitable $\tilde{k} \in \mathbb{N}$, the convex, lower semicontinuous function $\overline{\operatorname{conv}}(g^m)_{m\geq k}$ is finite at x and thus proper, see [2, Proposition 2.111]. Hence, we may use (2.2) such that, by means of (2.5),

$$(\inf_{m\geq k} g^m)^{**} \xrightarrow{e} g.$$

The continuity of the Legendre-Fenchel transform with respect to epi-convergence on the space of proper, convex, lower semicontinuous functions further gives

$$(\inf_{m\geq k} g^m)^{**} \xrightarrow{e} g \iff (\inf_{m\geq k} g^m)^* \xrightarrow{e} g^*,$$

see for example [10, Theorem 11.34]. By means of a standard argument, see [10, Proposition 7.4 (c)], the epi-limit of the decreasing sequence $(\inf_{m\geq k} g^m)^*$ is explicitly given by $\mathrm{cl}(\inf_k(\inf_{m\geq k} g^m)^*)$ such that

$$\mathrm{cl}(\inf_k(\inf_{m\geq k} g^m)^*) = g^*.$$

Recalling that the conjugate of any function g fulfills $g^*(0) = -\inf_{\mathbb{R}^n} g$, we obtain by assumption, along with [10, Theorem 11.23] as above, that

$$\mathrm{cl}(\inf_k(\inf_{m\geq k} g^m)^*)(0) = \inf_k(\inf_{m\geq k} g^m)^*(0) = \inf_k \sup_{m\geq k}(g^m)^*(0) =$$
$$\inf_k \sup_{m\geq k}(-\inf_{\mathbb{R}^n} g^m) = \inf_k(-\inf_{m\geq k}\inf_{\mathbb{R}^n} g^m) = -\sup_k \inf_{m\geq k}\inf_{\mathbb{R}^n} g^m.$$

Hence, $\lim_{k\to\infty}\inf_{m\geq k}\inf_{\mathbb{R}^n} g^m = \inf_{\mathbb{R}^n} g$. □

4 Minimal supersolutions of convex BSDEs

We now apply the results of the previous section in order to obtain stability theorems for minimal supersolutions of convex BSDEs. We start by introducing the stochastic framework and further notations.

4.1 Stochastic framework and notations

Given a time horizon $T > 0$, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and satisfies the usual conditions as well as $\mathcal{F}_T = \mathcal{F}$. We further denote with L^0 and L^0_t the set of $\mathcal{F}$-measurable and $\mathcal{F}_t$-measurable random variables, respectively, that are identified in the P-almost sure sense. In this connection, the sets L^p and L^p_t denote the set of elements in L^0 and L^0_t, respectively, with finite p-norm, for $p \in [1, +\infty]$. In addition, we introduce the notation $\mathcal{S} := \mathcal{S}(\mathbb{R})$ for the set of all càdlàg progressively measurable processes Y with values in $\mathbb{R}$, and we denote by $\mathcal{L}^p := \mathcal{L}^p(W)$, for $p \in [1, +\infty[$, the set of progressively measurable processes Z with values in $\mathbb{R}^d$ such that $||Z||_{\mathcal{L}^p} := E[(\int_0^T Z_s^2 ds)^{p/2}]^{1/p} < +\infty$. Hence, given $Z \in \mathcal{L}^p$, the stochastic integral $\int Z dW$ is well-defined and is a continuous martingale, see for instance [8]. In addition, we define with $\mathcal{L} := \mathcal{L}(W)$ the set of progressively measurable processes Z with values in $\mathbb{R}^d$, such that an increasing sequence $(\tau^k : \Omega \to [0,T])$ of stopping times with $P(\bigcup_k\{\tau^k = T\}) = 1$ exists, providing $Z1_{[0,\tau^k]} \in \mathcal{L}^1$, for all $k \in \mathbb{N}$. Here again, the stochastic integral $\int Z dW$ is well-defined and is a continuous local martingale. Throughout this section, unless otherwise stated, inequalities and strict inequalities between any two random variables or processes X^1, X^2 are understood in the P-almost sure or in the $P \otimes dt$-almost everywhere sense, respectively.

Throughout this chapter, a generator is a jointly measurable function g from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$ to $\mathbb{R} \cup \{\pm\infty\}$ where $\Omega \times [0,T]$ is endowed with the progressive σ-field. Moreover, $g(\omega, t, \cdot, \cdot)$ is assumed to be proper for all $(\omega, t) \in \Omega \times [0,T]$.

Let us define some properties of the generator that will be used in the sequel. We say a generator is

(LSC) if $(y, z) \mapsto g(y, z)$ is lower semincontinuous,

(POS) *positive* if $g(y, z) \geq 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

(CON) *convex* if $g(y, \lambda z + (1-\lambda)z') \leq \lambda g(y, z) + (1-\lambda) g(y, z')$, for all $y \in \mathbb{R}$, all $z, z' \in \mathbb{R}^d$, and all $\lambda \in (0,1)$, (INC) *increasing* if $g(y,z) \geq g(y',z)$, for all $y, y' \in \mathbb{R}$ with $y \geq y'$, and all $z \in \mathbb{R}^d$,

(DEC) *decreasing* if $g(y,z) \leq g(y',z)$, for all $y, y' \in \mathbb{R}$ with $y \geq y'$, and all $z \in \mathbb{R}^d$,

for all $(\omega, t) \in \Omega \times [0,T]$. Furthermore, we say that a sequence (g^k) of generators *converges pointwise* to a generator g, denoted by $g^k \xrightarrow{p} g$, if, for all $(\omega, t) \in \Omega \times [0,T]$, for all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $g^k(y,z) \to g(y,z)$.

4.2 Definitions

Given a *generator* g and a *terminal condition* $\xi \in L^0$, a *supersolution* of a BSDE is a pair $(Y,Z) \in \mathcal{S} \times \mathcal{L}$ that satisfies, for all $s,t \in [0,T]$ with $s \leq t$,

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi, \tag{4.1}$$

where Y is called the *value process* and Z its *control process*. In our setting, we additionally require Z to be *admissible*, that is, the continuous local martingale $\int ZdW$ is a supermartingale, in order to exclude *doubling-strategies*, see [7, Section 6.1] or [5]. Hence, we are interested in the set of supersolutions

$$\mathcal{A}(\xi, g) := \{(Y,Z) \in \mathcal{S} \times \mathcal{L} \ : \ Z \text{ is admissible and (4.1) holds}\}.$$

A supersolution (Y,Z) of $\mathcal{A}(\xi,g)$ is called *minimal* if, for any other element $(Y',Z') \in \mathcal{A}(\xi,g)$, and for all $t \in [0,T]$, it holds $Y_t \leq Y'_t$. Consider the process

$$\hat{\mathcal{E}}^g_t(\xi) := \operatorname{ess\,inf}\{Y_t \in L^0_t \ : \ (Y,Z) \in \mathcal{A}(\xi,g)\}, \quad t \in [0,T]. \tag{4.2}$$

Under the assumptions $\xi^- \in L^1$, $\mathcal{A}(\xi,g) \neq \emptyset$, and g fulfills (POS), (LSC), (CON) and either (INC) or (DEC), it can be shown that

$$\mathcal{E}^g_t(\xi) := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}^g_s(\xi), \text{ for all } t \in [0,T), \ \ \mathcal{E}^g_T(\xi) := \xi, \tag{4.3}$$

is a well defined càdlàg supermartingale, that it is a modification of $\hat{\mathcal{E}}^g(\xi)$, and that it is the value process of the unique minimal supersolution, that is there exists an admissible control process Z such that $(\mathcal{E}^g(\xi), Z) \in \mathcal{A}(\xi,g)$, see [4, Proposition 3.4.] and [4, Theorem 4.1.]. Under the preceding assumption on the generator and the terminal condition we will work with following convention. In case that $\mathcal{A}(\xi,g) = \emptyset$, we define $\hat{\mathcal{E}}^g(\xi)$ and $\mathcal{E}^g(\xi)$ as processes which are constant $+\infty$. Note that this and the fact that otherwise $\hat{\mathcal{E}}^g(\xi)$ is a modification of $\mathcal{E}^g(\xi)$ allows us to work exclusively with the notation $\mathcal{E}^g(\xi)$.

4.3 Stability

Given a generator g and a terminal condition ξ, we now address the stability of the minimal supersolution $\mathcal{E}^g(\xi)$ with respect to perturbations of the generator. As in [4] we will consider sequences of generators converging pointwise. However, in contrast to Drapeau et al. [4, Theorem 4.14] we do not assume that the sequence is monotone increasing. Given a sequence (g^k) of generators we will rely on the following regularization. We define, for all $k \in \mathbb{N}$, the functions $f^k, h^k : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \overline{\mathbb{R}}$ by

$$f^k(\omega,t,y,z) := \inf_{m \geq k} g^m(\omega,t,y,z), \tag{4.4}$$

and

$$h^k(\omega,t,y,z) := \overline{\operatorname{conv}}(g^m(\omega,t,y,\cdot))_{m \geq k}(z). \tag{4.5}$$

Lemma 4.1. *Let $k \in \mathbb{N}$. Assume that, for all $(\omega, t, y) \in \Omega \times [0,T] \times \mathbb{R}$,*

$$\operatorname{conv}(f^k(\omega,t,y,\cdot))(z) > -\infty, \tag{4.6}$$

for all $z \in \mathbb{R}^d$. Suppose further that, for all $k \le m \in \mathbb{N}$, for all $(\omega,t,y) \in \Omega \times [0,T] \times \mathbb{R}$, holds

$$\operatorname{int}\operatorname{dom}(g^m(\omega,t,y,\cdot)) \neq \emptyset \quad \textit{and} \quad \operatorname{int}\operatorname{dom}((f^k(\omega,t,y,\cdot))^*(\cdot)) \neq \emptyset. \tag{4.7}$$

Then h^k is a generator.

Proof. Due to (4.6) we may use the representation of h^k given by [10, Theorem 11.1 and Exercise 11.2], that is $h^k = (f^k)^{**}$ and where the dual operator is taken only with respect to the control variable z. In order to see that h^k is measurable we first show that

$$(\omega,t,y,z) \mapsto (f^k(\omega,t,y,\cdot))^*(z) \tag{4.8}$$

is measurable. We have

$$\begin{aligned}(f^k(\omega,t,y,\cdot))^*(z) &= \sup_{z^* \in \mathbb{R}^d} \{\langle z, z^*\rangle - f^k(\omega,t,y,z^*)\} \\ &= \sup_{m \ge k} \sup_{z^* \in \mathbb{R}^d} \{\langle z, z^*\rangle - g^m(\omega,t,y,z^*)\} \\ &= \sup_{m \ge k} \sup_{q^* \in \mathbb{Q}^d} \{\langle z, q^*\rangle - g^m(\omega,t,y,q^*)\},\end{aligned}$$

where the last inequality follows from convexity and lower semincontinuity of g^m in the control variable in combination with [10, Theorem 2.35] and Assumption (4.7).[1] Hence, since $(\omega,t,y,z) \mapsto \langle z, q^*\rangle - g^m(\omega,t,y,q^*)$ is measurable for every $q^* \in \mathbb{Q}$, it follows that (4.8) is measurable.

The same argument holds for the mapping $(\omega,t,y,z) \mapsto (f^k(\omega,t,y,\cdot))^{**}(z)$. Indeed, Assumptions (4.6) and (4.7) yield that the mapping $z \mapsto \operatorname{conv}(f^k(\omega,t,y,\cdot))(z))$ is proper. Hence, $z \mapsto (f^k(\omega,t,y,\cdot))^*(z)$ is convex and lower semincontinuous by [10, Theorem 11.1]. Now, Assumption (4.7) allows to conclude by the same reasoning as above that $h^k = (f^k)^{**}$ is measurable and hence a generator. □

The following Theorem is our main stability result.

Theorem 4.2. *Let $\xi \in L^0$ be a terminal condition such that $\xi^- \in L^1$, and let (g^k) be a sequence of generators which converges pointwise to a generator g. Suppose that each generator fulfills* (POS), (LSC), (CON) *and either* (INC) *or* (DEC). *Assume that the conditions of Lemma 4.1 are in force and that for all $(\omega,t) \in \Omega \times [0,T]$, the mapping $(y,z) \mapsto h^k(\omega,t,y,z)$ is lower semicontinuous. Finally, suppose that* $\operatorname{int}\operatorname{dom}(g(\omega,s,\cdot)) \neq \emptyset$ *as well as* $\operatorname{int}\operatorname{dom}((g(\omega,s,\cdot))^*) \neq \emptyset$*, for all $(\omega,s) \in \Omega \times [0,T]$. Then, we have $\mathcal{E}^g_0(\xi) \le \liminf_k \mathcal{E}^{g^k}_0(\xi)$. If, in addition $\lim \mathcal{E}^{h^k}_0(\xi) < +\infty$, then $\mathcal{A}(\xi, g) \neq \emptyset$, and, for all $t \in [0,T]$, it holds that $\mathcal{E}^g_t(\xi) \le \operatorname{ess\,lim\,inf}_k \mathcal{E}^{g^k}_t(\xi)$.*

Proof. For each $k \in \mathbb{N}$ the function h^k, defined in (4.5), is a generator by Lemma 4.1 and (LSC) by assumption. Moreover each h^k inherits (POS), (CON), and either (INC) or (DEC). Along with the conditions on the domain of g and g^* all assumptions of Theorem 3.1 are fulfilled and this yields

$$h^k \xrightarrow{p} g.$$

[1] Indeed, we can always find a sequence of rationals (q^n), such that $a := \inf_{z^* \in \mathbb{R}^d}\{g^m(\omega,t,y,z^*)\} = \lim_n g^m(\omega,t,y,q^n)$. To see that let $(z^n) \subset \mathbb{R}^d$ be such that $\lim_n g^m(z^n) = a$. Then, due to our assumptions on g^m and because of [10, Theorem 2.35] we can find, for each $n \in \mathbb{N}$, a rational q^n such that $|g^m(z^n) - g^m(q^n)| \le \frac{1}{n}$. Hence, $a = \limsup_n g^m(z^n) \ge \limsup_n (g^m(q^n) - \frac{1}{n}) = \limsup_n g^m(q^n) \ge \liminf_n g^m(q^n) \ge a$, that is $\lim_n g^m(q^n) = a$.

Hence, we obtain from [4, Proposition 3.2], for all $k \in \mathbb{N}$, and all $t \in [0,T]$,

$$\mathcal{E}_t^{h^k}(\xi) \leq \mathcal{E}_t^{g^k}(\xi) \text{ and } \mathcal{E}_t^{h^k}(\xi) \leq \mathcal{E}_t^{g}(\xi).$$

If $\lim \mathcal{E}_0^{h^k}(\xi) = +\infty$, then also $\liminf_k \mathcal{E}_0^{g^k}(\xi) = +\infty$ and hence

$$\mathcal{E}_0^{g}(\xi) \leq \liminf_k \mathcal{E}_0^{g^k}(\xi).$$

Suppose now $\lim \mathcal{E}_0^{h^k}(\xi) < +\infty$. From [4, Proposition 3.2] and (4.3) follows that, for all $k \in \mathbb{N}$, $\mathcal{A}(\xi, h^k) \neq \emptyset$ and $\mathcal{E}^{h^k}(\xi)$ is well-defined. This yields, for all $k \in \mathbb{N}$, and all $t \in [0,T]$,

$$\mathcal{E}_t^{h^k}(\xi) \leq \mathcal{E}_t^{h^k}(\xi) \leq \mathcal{E}_t^{g^k}(\xi).$$

Finally, we conclude with [4, Theorem 3.9] that $\mathcal{A}(\xi, g) \neq \emptyset$ and, for all $t \in [0,T]$,

$$\mathcal{E}_t^{g}(\xi) \leq \operatorname{ess}\liminf_k \mathcal{E}_t^{g^k}(\xi). \qquad \square$$

The following Proposition gives a sufficient condition for the joint lower semicontinuity of h^k as required in Theorem 4.2.

Proposition 4.3. *Let $k \in \mathbb{N}$. Assume that* (4.6) *holds and that, for all $(\omega, t) \in \Omega \times [0,T]$, the mapping $y \mapsto f^k(\omega,t,y,z)$ is lower semicontinuous, uniformly in $z \in \mathbb{R}^d$. Then the mapping $(y,z) \mapsto h^k(\omega,t,y,z)$ is lower semicontinuous. In particular, this is the case when f^k does not depend on y.*

Proof. As in the proof of Lemma 4.1 Condition (4.6) allows to use the representation of h^k given by $h^k = (f^k)^{**}$, where the dual operator is taken only with respect to the control variable z. Fix $z^* \in \mathbb{R}^d$. We show first that $y \mapsto (f^k(\omega,t,y,\cdot))^*(z^*)$ is upper semicontinuous. Let $(y^l) \subset \mathbb{R}$ be a sequence converging to $y \in \mathbb{R}$ and let $m := \limsup_l (f^k(\omega,t,y^l,\cdot))^*(z^*)$. The case $m = -\infty$ is obvious.

Assume $-\infty < m < \infty$. Then there exists, for $\varepsilon > 0$, a subsequence of (y^l), again denoted by (y^l), such that $m \leq (f^k(\omega,t,y^l,\cdot))^*(z^*) + \varepsilon < \infty$, for all $l \in \mathbb{N}$. By uniform lower semicontinuity we can choose $l_0 \in \mathbb{N}$ such that $-f^k(\omega,t,y^{l_0},z') \leq -f^k(\omega,t,y,z') + \varepsilon$, for all $z' \in \mathbb{R}^d$. Now choose $z \in \mathbb{R}^d$, such that $(f^k(\omega,t,y^{l_0},\cdot))^*(z^*) \leq \langle z^*, z\rangle - f^k(\omega,t,y^{l_0},z) + \varepsilon$. Hence,

$$\begin{aligned} m \leq (f^k(\omega,t,y^{l_0},\cdot))^*(z^*) + \varepsilon \leq \langle z^*, z\rangle - f^k(\omega,t,y^{l_0},z) + 2\varepsilon \\ \leq \langle z^*, z\rangle - f^k(\omega,t,y,z) + 3\varepsilon \leq (f^k(\omega,t,y,\cdot))^*(z^*) + 3\varepsilon, \end{aligned}$$

that is $m \leq (f^k(\omega,t,y,\cdot))^*(z^*)$.

Assume $m = \infty$. Then there exists a subsequence of (y^l), again denoted by (y^l), such that $(f^k(\omega,t,y^l,\cdot))^*(z^*)$ increases to infinity. Let $n \in \mathbb{N}$. There exists a subsequence of (y^l), again denoted by (y^l), such that $n+1 \leq (f^k(\omega,t,y^l,\cdot))^*(z^*)$, for all $l \in \mathbb{N}$. By uniform lower semicontinuity we can choose $l_0 \in \mathbb{N}$ such that $-f^k(\omega,t,y^{l_0},z') \leq -f^k(\omega,t,y,z') + \frac{1}{n}$, for all $z' \in \mathbb{R}^d$. Now choose $z \in \mathbb{R}^d$, such that $n \leq \langle z^*, z\rangle - f^k(\omega,t,y^{l_0},z)$. Hence,

$$n \leq \langle z^*, z\rangle - f^k(\omega,t,y^{l_0},z) \leq \langle z^*, z\rangle - f^k(\omega,t,y,z) + \frac{1}{n} \leq (f^k(\omega,t,y,\cdot))^*(z^*) + \frac{1}{n}$$

Sending n to infinity implies $m = (f^k(\omega,t,y,\cdot))^*(z^*)$.

We conclude that $y \mapsto (f^k(\omega,t,y,\cdot))^*(z^*)$ is upper semicontinuous and that

$$(y,z) \mapsto h^k(\omega,t,y,z) = (f^k(\omega,t,y,\cdot))^{**}(z) = \sup_{z^*}\{\langle z, z^*\rangle - (f^k(\omega,t,y,\cdot))^*(z^*)\}$$

is lower semicontinuous as supremum of lower semicontinuous functions, see for example [10, Proposition 1.26]. $\square$

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