Portfolio Optimization under Nonlinear Utility

Gregor Heyne[1,*], Michael Kupper[a,2], Ludovic Tangpi[a,3,†]

April 15, 2015

ABSTRACT

This paper studies the utility maximization problem of an agent with non-trivial endowment, and whose preferences are modeled by the maximal subsolution of a BSDE. We prove existence of an optimal trading strategy and relate our existence result to the existence of a maximal subsolution to a controlled decoupled FBSDE. Using BSDE duality, we show that the utility maximization problem can be seen as a robust control problem admitting a saddle point if the generator of the BSDE additionally satisfies a specific growth condition. We show by convex duality that any saddle point of the robust control problem agrees with a primal and a dual optimizer of the utility maximization problem, and can be characterized in terms of a BSDE solution.

KEYWORDS: Subsolutions of BSDEs; submartingale; Convex duality; Utility maximization.

AUTHORS INFO

[a] Universität Konstanz, Universitätsstraße 10, 78464 Konstanz, Germany

[1] gh.heyne@gmail.com

[2] kupper@uni-konstanz.de

[3] ludovic.tangpi@uni-konstanz.de

[*] MATHEON project E.2

[†] Partially supported by a Ph.D. scholarship of the Berlin Mathematical School

PAPER INFO

AMS CLASSIFICATION: 60H20; 93E20; 91B16; 91G10

1. Introduction

The theory of expected utility is of fundamental importance in finance and economy. Introduced by Bernoulli [2], the expected utility represents the level of satisfaction of a financial agent acting in a risky environment. In their seminal *Theory of Games and Economic Behavior*, von Neumann and Morgenstern [32] have provided an axiomatic foundation for decision making under risk based on *rational* principles; and by the work of Savage [31], under these axioms preferences can be modeled as expected utility. However, the axioms of von Neumann and Morgenstern have been much criticized by empirical studies such as the well known Allais paradox and Ellsberg paradox. On the other hand, expected utility does not capture uncertainty in the underlying probabilistic model. Many alternative approaches have been suggested to model decision beyond expected utility. A few examples include the concepts of capacity and weighted expected utility and, more recently, the recursive utility and the g-expectation. Following this trend, we consider in the present work the portfolio optimization of an agent whose utility is modeled by the maximal subsolution of a nonlinear backward stochastic differential equation (BSDE). Our principal aim is to give sufficient, and necessary conditions of existence of an optimal portfolio in this framework.

Amongst the numerous attempts that have been made in the literature to study portfolio optimization under nonlinear utility, the work of El Karoui et al. [17] on the optimization of stochastic differential utility is especially related to ours. This class of utility functions were introduced by Duffie and Epstein [14] and can be seen as solutions of nonlinear BSDEs. In a non-Markovian model, El Karoui et al. [17] prove existence of an optimal trading strategy and an optimal consumption policy and characterize the optimal wealth process and the utility as solutions of a forward-backward system. They assume that the generator of the BSDE satisfies a linear growth condition and is continuously differentiable in all variables, so that the utility itself is differentiable and satisfies a comparison principle. Their results are based on BSDE theory: Notably, the existence result follows from a penalization method which consists in approaching the problem by a sequence of penalized problems that can be solved, and then obtain the solution by compactness arguments.

☆We thank Victor Nzengang for fruitful discussions

The first contribution of the present paper is to give conditions that guarantee the existence of an optimal trading strategy for an agent whose utility is given as the maximal subsolution of a BSDE. We consider a non-Markovian incomplete market model where the agent also has a random terminal endowment, and the utility is modeled by a BSDE whose generator is convex, positive, lower semicontinuous and satisfies a normalization condition. The technique of the proof, inspired from Drapeau et al. [12], rests on localization arguments and compactness principles. We do not impose any artificial integrability with respect to the historical probability measure on the wealth process. Hence, the central idea here is to introduce an auxiliary function under which the image of the terminal conditions will be uniformly integrable in the set of subsolutions. To this end, we require the drift to satisfy a suitable integrability condition. This uniform integrability allows for the construction of a localizing sequence of stopping times that makes the value processes of the admissible subsolutions local submartingales. Thus, compactness results for sequences of martingales, see Delbaen and Schachermayer [7], and sequences of increasing finite variation processes can be used locally in time, and the candidate solutions obtained by almost sure convergence of the sequence of stopping times to the time horizon. The verification follows from Fatou's lemma and join convexity of the generator.

Analogous to the case of recursive utility studied by El Karoui et al. [17], there is an intrinsic link between the optimal wealth process and its utility: They can be seen as a maximal subsolution of a forward-backward system.

We also address the question of characterization of an optimal trading strategy. In the optimal stochastic control literature, such a characterization is usually a consequence of the stochastic maximum principle. One introduces a perturbation of the optimal diffusion and, by Itô's formula, obtains at the limit a variational equation which enables to characterize the optimal control, see for instance Peng [26] and Horst et al. [20]. This characterization follows from the fact that the expectation operator is linear, a property that our operator does not enjoy. The idea to get around this difficulty is to use the duality of BSDEs studied by Drapeau et al. [13], and transform the original control problem into a robust control problem with non-zero penalty term. Provided that the robust control problem admits a saddle point, the problem can be linearized and the maximum principle applies. The proof of the existence of a saddle point follows from the existence of an optimal trading strategy and a weak compactness argument introduced by Delbaen et al. [9] which is achieved under a growth condition on the generator of the BSDE.

The theory of BSDE duality fits quite well to our setting. It shows for instance that our maximization problem is nothing but the maximization of recursive utilities under model uncertainty. And because our generator depends on the value process, the uncertainty here also encompasses the uncertainty about the time value of money, see El Karoui and Ravanelli [16] and Drapeau et al. [13]. It also enables us to write and solve the dual problem and characterize its solution in terms of solutions of a BSDE, and shows that the dual optimizer is, in fact, the optimal probabilistic model.

Before presenting the structure of our work, let us give further references of related works. Using a convex duality approach, the expected utility maximization problem was studied by Kramkov and Schachermayer [24]. They give precise conditions on the utility function for a solution to exist. Cvitanić et al. [4] have extended their results to the non-zero random endowment case. A fully probabilistic method to study the problem has been investigated by Hu et al. [21]. For exponential utility, they characterize the value function and the optimal strategy of the problem with random endowment as the solution of a quadratic BSDE. Beyond the exponential utility case, Horst et al. [20] show that the problem can be solved via forward backward systems. Robust expected utilities have been considered by Bordigoni et al. [3] and Faidi et al. [18]. The latter authors consider a problem with non-zero penalty term and prove existence of and optimal model. Øksendal and Sulem [25] show that the robust control problem can be treated as a stochastic differential game, a consideration that is also implicitly used in the present paper.

The next section of the paper is dedicated to the setting of the probabilistic framework of our study and introduces the market model. Section 3 studies the primal problem: We prove existence of an optimal strategy and stability of the utility operator. The third section deals with the dual problem. Notably, we prove existence of a dual optimizer and characterize the dual and primal optimizers by means of BSDE solutions. In the last section, we draw the link between duality of BSDEs and the general theory of convex duality. We gather in an appendix some proofs that are classical in the theory of convex BSDEs but still need to be adapted to our setting for completeness.

2. Setup and Market Model

Let $T \in (0, \infty)$ be a fixed time horizon, and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space. The filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and satisfies the usual assumptions of completeness and right-continuity, with $\mathcal{F}_T = \mathcal{F}$. Statements concerning random variables or stochastic processes are understood in the P-almost sure or the $P \otimes dt$-almost sure sense, respectively. Indistinguishable processes are identified. When we make a statement without any precision regarding the probability measure, then we are referring to the probability measure P. Thus, by "M is a martingale" we mean "M is a P-martingale".

We write L^0 for the space of $\mathcal{F}$-measurable random variables endowed with the topology of convergence in probability with respect to the measure P. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of adapted processes with values in $\mathbb{R}$ which are càdlàg. For $p \in [1, \infty]$, the space $L^p(\Omega, \mathcal{F}, P)$ is denoted by L^p and for a different measure Q we write $L^p(Q)$ for $L^p(\Omega, \mathcal{F}, Q)$. The space L^p_+ is the space of positive random variables belonging to L^p. We further denote by $\mathcal{L}^p := \mathcal{L}^p(P)$ the set of predictable processes Z with values in $\mathbb{R}^{1\times d}$, endowed with the norm $\|Z\|_{\mathcal{L}^p} := E_P[(\int_0^T \|Z_s\|^2\, ds)^{p/2}]^{1/p}$. From [27], for every $Z \in \mathcal{L}^p$ the process $(\int_0^t Z_s\, dW_s)_{t\in[0,T]}$ is well defined and by means of Burkholder-Davis-Gundy's inequality, it is a continuous martingale. By $\mathcal{L}$ we denote the set of predictable processes valued in $\mathbb{R}^{1\times d}$ such that there exits a localizing sequence of stopping times (τ^n) with $Z1_{[0,\tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. For $Z \in \mathcal{L}$, the stochastic process $(\int_0^t Z_u\, dW_u)_{t\in[0,T]}$ is a well defined continuous local martingale. Furthermore, for adequate integrands a and Z we write $\int a\, ds$ and $\int Z\, dW$ for $(\int_0^t a_s\, ds)_{t\in[0,T]}$ and $(\int_0^t Z_u\, dW_u)_{t\in[0,T]}$, respectively. The running maximum of a process X is denoted by $X^*_t = \sup_{s\in[0,t]} |X_s|$. Given a sequence (x_n) in some convex set, a sequence $(\tilde{x}_n)$ is said to be in the asymptotic convex hull of (x_n) if $\tilde{x}_n \in conv\{x^n, x^{n+1}, \dots\}$ for all n.

In the financial market, there are available for trading n stocks, $n \leq d$, with price dynamics

$$dS^i_t = S^i_t(\mu^i_t\, dt + \sigma^i_t\, dW_t), \quad i = 1, \dots, n,$$

such that μ^i and σ^i are predictable processes valued in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. Let us denote by σ the $n \times d$ matrix with row vectors σ^i, the matrix[1] $\sigma\sigma'$ is assumed to be of full rank, so that the market price of risk θ takes the form $\theta_t = \sigma'_t(\sigma_t\sigma'_t)^{-1}\mu_t$, $t \in [0, T]$. For the rest of the paper, we make the following standing assumption concerning θ:

- There exist constants $p > 1$ and $C_\theta > 0$ such that for all stopping times $0 \leq \tau \leq T$, one has

$$E\left[\left(\mathcal{E}\left(\int \theta\, dW\right)_\tau / \mathcal{E}\left(\int \theta\, dW\right)_T\right)^{\frac{1}{p-1}} \mid \mathcal{F}_\tau\right] \leq C_\theta, \tag{A}$$

where $\mathcal{E}(\int \theta\, dW)$ denotes the stochastic exponential of $\int \theta\, dW$. This is the so-called Muckenhoupt A_p condition. Under this assumption, by [23, Theorem 2.4], $\int \theta\, dW$ is a BMO martingale, and therefore $\frac{dQ}{dP} = \mathcal{E}(-\int \theta\, dW)_T$ defines a probability measure Q equivalent to P. This type of drift conditions are well-known, especially in the context of expected utility maximization, see for instance Delbaen et al. [8]. Let $x > 0$ be a fixed initial capital. A trading strategy is a predictable d-dimensional process π such

[1] σ' is the transpose of σ.

that $\pi\sigma \in \mathcal{L}(Q)$ and $X^\pi \geq 0$, where the wealth process X^π is given by

$$X_t^\pi = x + \int_0^t \pi_s \sigma_s(\theta_s \, ds + dW_s), \quad t \in [0, T]. \tag{2.1}$$

We denote by Π the set of trading strategies. For every $\pi \in \Pi$, X^π is a positive Q-local martingale and thus a Q-supermartingale. In particular, the market is free of arbitrage opportunities. The principal objective of this paper is to study the utility maximization from the terminal wealth of an agent who has a non-trivial endowment ξ and whose utility is modeled by a BSDE.

The generator we consider for the BSDEs is a jointly measurable function $g : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1\times d} \to \mathbb{R} \cup \{+\infty\}$, where $\Omega \times [0, T]$ is endowed with the predictable σ-algebra. Furthermore, a generator g is said to be

(CONV) convex, if $(y, z) \mapsto g(y, z)$ is convex,

(LSC) lower semicontinuous, if $(y, z) \mapsto g(y, z)$ is lower semicontinuous,

(NOR) normalized, if $g(y, 0) = 0$ for all $y \in \mathbb{R}_+$,

(POS) positive, if $g \geq 0$.

Given a random variable $H \in L^0$, a subsolution of the BSDE with generator g and terminal condition H is a pair (Y, Z) of processes satisfying

$$Y_s + \int_s^t g_u(Y_u, Z_u) \, du - \int_s^t Z_u \, dW_u \leq Y_t; \quad Y_T \leq H, \tag{2.2}$$

for all $0 \leq s \leq t \leq T$. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a continuous concave, strictly increasing function such that there exists $C > 0$, $|u(x)|^{p^2} \leq C(1 + |x|)$ for every $x > 0$, with p introduced in the condition (A) and such that $L \mapsto u^{-1}(E[u(L)])$ is concave on $\{L \in L^0_+ : E[u(L)] < +\infty\}$. Examples of such a function include $u(x) = x^r$ with $rp^2 < 1$, and $u(x) = -\exp(-rx)$ with $r > 0$, see [11, Section 3].

A value process $Y \in \mathcal{S}_+$ is said to be admissible if the process $u(Y)$ is a submartingale. We consider the operator

$$\mathcal{E}_0^g(H) := \sup \{Y_0 : (Y, Z) \in \mathcal{A}^u(H, g)\}$$

with

$$\mathcal{A}^u(H, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Y \text{ admissible and (2.2) holds}\},$$

the set of admissible subsolutions with respect to u. The reader will notice that the operator $\mathcal{E}_0^g(\cdot)$ depends on u. Similar to [12] the operator $\mathcal{E}_0^g(\cdot)$ is a nonlinear utility function. In particular, it is monotone, concave but not necessarily cash-additive. We study the investment problem

$$V(x) := \sup_{\pi\in\Pi} \mathcal{E}_0^g(\xi + X_T^\pi). \tag{2.3}$$

More precisely, we would like to give conditions of existence of a pair $(\bar{Y}, \bar{Z})$ along with a trading strategy $\bar{\pi} \in \Pi$ such that $(\bar{Y}, \bar{Z}) \in \mathcal{A}^u(\xi + X_T^{\bar{\pi}}, g)$ and for any other trading strategy $\pi \in \Pi$ one has

$$\bar{Y}_0 = V(x) = \mathcal{E}_0^g(\xi + X_T^{\bar{\pi}}) \geq \mathcal{E}_0^g(\xi + X_T^\pi).$$

Henceforth, the function V will be referred to as the value function of the optimization problem (2.3), and the triple $(\bar{X}, \bar{Y}, \bar{Z})$ with $\bar{X} = X^{\bar{\pi}}$, a maximal subsolution.

Example 2.1. 1. Certainty equivalent: Let X be an $\mathcal{F}_T$-measurable random variable such that $u(X)$ is integrable. The certainty equivalent $C_t(X)$ of X is defined as $C_t(X) := u^{-1}\left(E[u(X)\,|\,\mathcal{F}_t]\right)$, $t \in [0,T]$. Consider the utility maximization problem

$$V(x) = \sup_{\pi\in\Pi} C_0(X_T^\pi + \xi). \tag{2.4}$$

The martingale representation theorem yields a process $N \in \mathcal{L}^1$ such that

$$E[u(X)\,|\,\mathcal{F}_t] = E[u(X)] + \int_0^t N_u\,dW_u, \quad \text{for all } t \in [0,T].$$

Applying Itô's formula to $Y_t = u^{-1}\left(E[u(X)\,|\,\mathcal{F}_t]\right)$, we have

$$dY_t = \frac{1}{u'(Y_t)} N_t\,dW_t - \frac{1}{2}\frac{u''(Y_t)}{(u'(Y_t))^3}\left|N_t\right|^2\,dt.$$

Hence, putting $Z_t = \frac{1}{u'(Y_t)}N_t$, the pair (Y,Z) solves the BSDE

$$Y_t = X + \frac{1}{2}\int_t^T \frac{u''(Y_u)}{u'(Y_u)}\left|Z_u\right|^2\,du - \int_t^T Z_u\,dW_u. \tag{2.5}$$

For $u(x) = x^r$, $r \in (0,1)$, the generator of the BSDE (2.5) is given by $g(y,z) = \frac{1}{2}(r-1)\left|z\right|^2/y$ and satisfies the conditions (CONV), (LSC), (NOR) and (POS) on $(0,+\infty)\times\mathbb{R}^d$. By definition, we have $\mathcal{E}_0^g(X) \geq C_0(X)$. In addition, the admissibility condition implies $u(\mathcal{E}_0^g(X)) \leq E\left[u(\mathcal{E}_T^g(X))\right]$. Therefore, $\mathcal{E}_0^g(X) \leq C_0(X)$. Thus, the utility maximization problem (2.4) can be rewritten as $V(x) = \sup_{\pi\in\Pi}\mathcal{E}_0^g(X_T^\pi+\xi)$.

2. g-expectation: Let u be a utility function and g a function defined on $\mathbb{R}\times\mathbb{R}^d$ and satisfying (LSC), (NOR) and (POS) such that for every $\pi\in\Pi$ the BSDE with terminal condition $u(X_T^\pi+\xi)$ and generator g has a unique solution $(Y^\pi, Z^\pi) \in \mathcal{S}\times\mathcal{L}^2$. Denote by $\mathcal{E}_g[u(X_T^\pi+\xi)\,|\,\mathcal{F}_t] := Y_t^\pi$ the g-expectation of $u(X_T^\pi+\xi)$. The operator $\mathcal{E}_g[\cdot]$ is a nonlinear expectation which coincides with the classical expectation $E_P[\cdot]$ when $g=0$. Consider the utility maximization problem

$$V(x) := \sup_{\pi\in\Pi} u^{-1}\left(\mathcal{E}_g[u(\xi + X_T^\pi)\,|\,\mathcal{F}_0]\right).$$

We further assume u to be twice continuously differentiable and that u' is bounded away from zero. For every $\pi\in\Pi$, we have

$$Y_t^\pi = u(X_T^\pi+\xi) + \int_t^T g(Y_u^\pi, Z_u^\pi)\,du - \int_t^T Z_u^\pi\,dW_u. \tag{2.6}$$

Applying Itô's formula to $\hat{Y}_t^\pi := u^{-1}(Y_t^\pi)$, we obtain

$$d\hat{Y}_t^\pi = -\left\{\frac{1}{u'(\hat{Y}_t^\pi)} g(u(\hat{Y}_t^\pi), \hat{Z}_t^\pi u'(\hat{Y}_t^\pi)) - \frac{1}{2}\frac{u''(\hat{Y}_t^\pi)}{u'(\hat{Y}_t^\pi)}\left|\hat{Z}_t^\pi\right|^2\right\}dt + \hat{Z}_t^\pi\,dW_t, \tag{2.7}$$

with $\hat{Z}_t^\pi = Z_t^\pi/u'(Y_t^\pi)$ and $\hat{Y}_T^\pi = X_T^\pi+\xi$. For $u(x) = -\exp(-rx)$, $r>0$ and $g(y,z) = |z|$, the generator of the above BSDE takes the form $\hat{g}(y,z) = |z| + \frac{1}{2}(1-r)r^2\left|z\right|^2$ and it satisfies the properties (CONV), (LSC), (NOR) and (POS). Since g is positive, Y^π is a submartingale and we have $\mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi) \ge \hat{Y}_0^\pi = u^{-1}(Y_0^\pi) = u^{-1}(\mathcal{E}_g(u(X_T^\pi+\xi)\,|\,\mathcal{F}_0))$. In addition, the admissibility condition implies $u(\mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi)) \le E[u(\mathcal{E}_T^{\hat{g}}(X_T^\pi+\xi))] \le E[u(X_T^\pi+\xi)]$ by monotonicity of u. Since g is positive, taking expectation of both sides of (2.6) yields $\mathcal{E}_g(u(X_T^\pi+\xi)\,|\,\mathcal{F}_0) \ge E[u(X_T^\pi+\xi)]$. Therefore, $\mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi) \le u^{-1}(\mathcal{E}_g(u(X_T^\pi+\xi)\,|\,\mathcal{F}_0))$. Thus, the utility maximization problem (2.4) can be rewritten as $V(x) = \sup_{\pi\in\Pi} \mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi)$. $\diamondsuit$

3. Maximal Subsolutions

3.1. Existence Results

In this section we give sufficient conditions of existence of an optimal trading strategy to Problem (2.3). In order to simplify the presentation, let us introduce the set

$$\mathcal{A}(x) := \{(X,Y,Z) : X \text{ satisfies (2.1) for some } \pi \in \Pi \text{ and } (Y,Z) \in \mathcal{A}^u(\xi + X_T, g)\}.$$

The function $V(x)$ can be written as

$$V(x) = \sup\{Y_0 : (X,Y,Z) \in \mathcal{A}(x)\}.$$

If g satisfies (NOR) and $\xi \ge 0$, the set $\mathcal{A}(x)$ is nonempty, and contains an element with positive value process. The triplet (X^0, Y^0, Z^0), with $Z^0 = 0$, $Y^0 = X^0 = x$ and with associated trading strategy $\pi = 0$ is an element of $\mathcal{A}(x)$. Indeed, the pair (Y^0, Z^0) satisfies (2.2), and we have $Y_T^0 = x \le x + \xi = \xi + X_T^0$. Moreover, for all $(X,Y,Z) \in \mathcal{A}(x)$ the càdlàg process Y can jump only up, since by taking the limit as s tends to $t-$ in Equation (2.2) we have $Y_t \ge Y_{t-}$, for all $t \in [0,T]$. Before stating our existence result, let us prove the following lemmas.

Lemma 3.1. *Assume $\xi \in L^1_+(\Omega, \mathcal{F}_T, Q)$. Then there exists a constant $C \ge 0$ such that for all $(X,Y,Z) \in \mathcal{A}(x)$ with $Y \ge 0$, we have*

$$E[|u(\xi+X_T)|^p] \le C \quad \text{and} \quad u(Y_t) \le E[u(\xi + X_T)\,|\,\mathcal{F}_t] \quad t \in [0,T].$$

Proof. Let (X,Y,Z) be in $\mathcal{A}(x)$, and q the Hölder conjugate of p. We first prove the L^p boundedness of $u(\xi+X_T)$. Using Hölder's inequality, we estimate as follows:

$$\begin{aligned} E[|u(\xi+X_T)|^p] &= E_Q\left[\frac{1}{\mathcal{E}(\int\theta\,dW)_T}|u(\xi+X_T)|^p\right] \\ &\le E_Q\left[\left(\frac{1}{\mathcal{E}(\int\theta\,dW)_T}\right)^q\right]^{\frac{1}{q}} E_Q[|u(\xi+X_T)|^{p^2}]^{\frac{1}{p}}. \end{aligned}$$

Since there exists a positive constant C such that

$$|u(\xi+X_T)|^{p^2} \le C(1+\xi+X_T),$$

we have

$$E[|u(\xi+X_T)|^p] \le C^{1/p} E\left[\mathcal{E}\left(\int\theta\,dW\right)_T\left(\frac{1}{\mathcal{E}(\int\theta\,dW)_T}\right)^q\right]^{\frac{1}{q}} E_Q[1+\xi+X_T]^{\frac{1}{p}}.$$

Thus, since $q-1 = \frac{1}{p-1}$, it follows from the Muckenhoupt A_p condition and the Q-supermartingale property of X, that

$$E\left[|u(\xi+X_T)|^p\right] \le C^{1/p} C_\theta^{1/q} (1 + E_Q[\xi] + x)^{\frac{1}{p}},$$

hence the first estimate.

For the second estimate, first notice that $u(\xi + X_T)$ is integrable, and since u is increasing and (Y, Z) satisfies Equation (2.2), we have $u(Y_T) \le u(\xi + X_T)$. Since the value process Y is admissible, we have $u(Y_t) \le E[u(Y_T)\,|\,\mathcal{F}_t] \le E[u(\xi + X_T)\,|\,\mathcal{F}_t]$ for all $t \in [0, T]$. □

The previous lemma gives two *a priori* estimates for subsolutions of Equation (2.2). In particular, it shows that the family of random variables $u(\xi + X_T)$, when (X, Y, Z) runs through $\mathcal{A}(x)$, is uniformly integrable.

Remark 3.2. a) Due to the admissibility condition and the previous lemma, it holds $V(x) \in \mathbb{R}$ for every $x > 0$. In fact, for any $(X, Y, Z) \in \mathcal{A}(x)$, since $(x, x, 0) \in \mathcal{A}$, we can assume $Y_0 \ge x$. By admissibility,

$$u(Y_0) \le E[u(Y_T)] \le E[u(\xi + X_T)].$$

Lemma 3.1 and Jensen's inequality give

$$u(Y_0)^p \le E[|u(\xi + X_T)|^p] \le C.$$

b) If a subsolution $(X, Y, Z) \in \mathcal{A}(x)$ is such that $\log(Y)$ is a submartingale, then since $Y_0 \ge x$, we have $E[\log(Y_t)] \ge \log(x) > 0$ for all $t \in [0, T]$. Hence, $Y_t = 0$ with probability zero. Therefore, the function $u = \log$ can be used to defined admissibility of subsolutions. ♦

The next lemma describes the set of subsolutions.

Lemma 3.3. *If g satisfies (CONV), then the set $\mathcal{A}(x)$ is convex.*

Proof. See Appendix A. □

The following existence theorem is the first main result of this paper.

Theorem 3.4. *Assume that the generator g satisfies (CONV), (LSC), (NOR) and (POS); and that the random endowment ξ belongs to L^∞_+. Then there exists a trading strategy $\bar{\pi} \in \Pi$ with associated wealth process $\bar{X}$ and a pair $(\bar{Y}, \bar{Z}) \in \mathcal{A}^u(\xi + \bar{X}_T, g)$ such that $\bar{Y}_0 = V(x)$.*

Proof. Let $((X^n, Y^n, Z^n))$ be a sequence in $\mathcal{A}(x)$ such that $Y_0^n \uparrow V(x)$. The proof goes in several steps. We start by making some transformations on the maximizing sequence $((X^n, Y^n, Z^n))$.

Step 1 Preliminary transformations. The sequence $((X^n, Y^n, Z^n))$ can be considered to be such that for all $n \in \mathbb{N}$, $Y_0^n \ge x$ and $Y^n \ge X^n$. In fact, since the set $\mathcal{A}(x)$ contains the triple $(x, x, 0)$, by definition of $V(x)$ it holds $V(x) \ge x$. Hence, we can assume without loss of generality that $Y_0^n \ge x$, for all n. For each $n \in \mathbb{N}$, define the stopping time δ^n by

$$\delta^n := \inf\{t \ge 0 : Y_t^n \le X_t^n\} \wedge T,$$

and put

$$\hat{Y}^n := Y^n 1_{[0,\delta^n)} + Y^n_{\delta^n} 1_{[\delta^n, T]}; \quad \hat{Z}^n := Z^n 1_{[0,\delta^n]}$$

and

$$\hat{X}^n := X^n 1_{[0,\delta^n]} + X^n_{\delta^n} 1_{[\delta^n, T]}.$$

The triple $(\hat{X}^n, \hat{Y}^n, \hat{Z}^n)$ belongs to $\mathcal{A}(x)$. In fact, for all $s, t \in [0,T]$ with $0 \le s \le t \le T$, on the set $\{s \le \delta^n \le t\}$ we have

$$\begin{aligned}
\hat{Y}^n_s &+ \int_s^t g_u(\hat{Y}^n_u, \hat{Z}^n_u)\, du - \int_s^t \hat{Z}^n_u \, dW_u \\
&= Y^n_s + \int_s^{\delta^n} g_u(Y^n_u, Z^n_u)\, du - \int_s^{\delta^n} Z^n_u \, dW_u + \int_{\delta^n}^t g_u(Y^n_{\delta^n}, 0)\, du \\
&\le Y^n_{\delta^n} = \hat{Y}^n_{\delta^n}.
\end{aligned}$$

On the sets $\{s \ge \delta^n\}$ and $\{t \le \delta^n\}$ the proof is the same. Now for the forward process, let $t \in [0,T]$. On the set $\{\delta^n \le t\}$, putting $\hat{\pi}^n := \pi^n 1_{[0,\delta^n]}$, we have

$$\hat{X}^n_t = X^n_{\delta^n} = x + \int_0^{\delta^n} X^n_u \pi^n_u \sigma_u \, dW^Q_u + \int_{\delta^n}^t 0 \, dW^Q_u = x + \int_0^t \hat{X}^n_u \hat{\pi}^n_u \sigma_u \, dW^Q_u.$$

On $\{t \le \delta^n\}$ there is nothing to prove. In order to show that the terminal condition is satisfied, notice that on the set $\{\delta^n < T\}$ it holds $Y^n_{\delta^n} = X^n_{\delta^n}$. This is because $Y^n_0 \ge x$, X^n is continuous and Y^n only jumps upward. Thus,

$$\hat{Y}^n_T = Y^n_{\delta^n} = X^n_{\delta^n} \le X^n_{\delta^n} + \xi = \hat{X}^n_T + \xi$$

and on the set $\{\delta^n = T\}$ it holds

$$\hat{Y}^n_T = Y^n_T \le \xi + X^n_T = \xi + \hat{X}^n_T.$$

In addition, for all $n \in \mathbb{N}$, $\hat{\pi}^n$ is a trading strategy and $u(\hat{Y}^n)$ is a P-submartingale. In fact, for all $0 \le s \le t \le T$, due to the admissibility of Y^n, we have

$$E[u(\hat{Y}^n_t) - u(\hat{Y}^n_s) \,|\, \mathcal{F}_s] = E[u(Y^n_{(s \vee \delta^n) \wedge t}) - u(Y^n_s) \,|\, \mathcal{F}_s] \ge 0.$$

Hence $\hat{Y}^n$ is admissible. Therefore, we have

$$((\hat{X}^n, \hat{Y}^n, \hat{Z}^n)) \subseteq \mathcal{A}(x)$$

with $\hat{Y}^n_0 \uparrow V(x)$ and for all $t \in [0,T]$, $\hat{X}^n_t \le \hat{Y}^n_t$. In the sequel of the proof we shall simply write (X^n, Y^n, Z^n) for $(\hat{X}^n, \hat{Y}^n, \hat{Z}^n)$, for every $n \in \mathbb{N}$.

Step 2 An estimate for the value process. Now we provide a bound on the value process that will be a key ingredient for the localization in the subsequent step. Since (X^n_T) is a sequence of positive random variables, by [6, Lemma A1.1] there exists a sequence denoted $(\tilde{X}^n_T)$ in the asymptotic convex hull of (X^n_T) and an $\mathcal{F}_T$-measurable random variable X such that

$$\lim_{n\to\infty} \tilde{X}^n_T = X \quad Q\text{-a.s.}$$

Let $(\tilde{X}^n)$ be the sequence in the asymptotic convex hull associated to $(\tilde{X}^n_T)$. For each $n \in \mathbb{N}$ the process $\tilde{X}^n$ is positive and inherits the Q-supermartingale property of X^n, that is, $E_Q[\tilde{X}^n_T] \le x$. Hence, it follows from Fatou's lemma that

$$x \ge \liminf_{n\to\infty} E_Q[\tilde{X}^n_T] \ge E_Q[\liminf_{n\to\infty} \tilde{X}^n_T] = E_Q[X].$$

By continuity of the function u and Q-almost sure convergence of $(\tilde{X}^n_T)$ it follows that $(u(\xi+\tilde{X}^n_T))$ converges to $u(\xi+X)$ Q-a.s., and therefore P-a.s. by equivalence of measures. Moreover, due to Lemmas 3.3 and 3.1, the family $(u(\xi+\tilde{X}^n_T))_n$ is uniformly integrable. Therefore, we can conclude using the dominated convergence theorem that

$$\lim_{n\to\infty} u(\xi+\tilde{X}^n_T) = u(\xi+X) \quad \text{in } L^1. \tag{3.1}$$

For all $n\in\mathbb{N}$ and $t\in[0,T]$ define

$$M^n_t := E[u(\xi+\tilde{X}^n_T)\,|\,\mathcal{F}_t] \quad \text{and} \quad M_t := E[u(\xi+X)\,|\,\mathcal{F}_t].$$

We denote by $((\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n))$ the sequence in the asymptotic convex hull of $((X^n,Y^n,Z^n))$ associated to $(\tilde{X}^n_T)$. By Lemma 3.3, $((\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n))\subseteq\mathcal{A}(x)$, and Lemma 3.1 leads to

$$u(\tilde{Y}^n_t)\le M^n_t\le (M^n_\cdot)^*_T \quad \text{for all } t\in[0,T],$$

which implies, since u^{-1} is increasing, that $\tilde{Y}^n_t\le u^{-1}((M^n_\cdot)^*_T)$. Thus, $(\tilde{Y}^n_\cdot)^*_T\le u^{-1}((M^n_\cdot)^*_T)$; recall that $\tilde{Y}^n_t\ge\tilde{X}^n_t\ge 0$. Using again the fact that u^{-1} is increasing and the inequalities

$$(M^n_\cdot)^*_T\le (M^n_\cdot - M_\cdot + M_\cdot)^*_T\le (M^n_\cdot - M_\cdot)^*_T + M^*_T$$

we finally have

$$(\tilde{Y}^n_\cdot)^*_T\le u^{-1}((M^n_\cdot - M_\cdot)^*_T + M^*_T).$$

Step 3 Local bound for the control process. Here we obtain an estimate that will enable us to use a compactness argument for the space $\mathcal{L}^1$. That estimate stems from the fact that Y^n can be shown to be a local submartingale. We start by introducing a localization of the value processes. Since the sequence (M^n_T) converges in L^1, for a given $k\in\mathbb{N}$ we may, and do, choose a subsequence $(M^{n,k})_n$ such that

$$E[|M^{n,k}_T - M_T|]\le\frac{2^{-n}}{k} \quad n\in\mathbb{N}. \tag{3.2}$$

Let $((\tilde{X}^{n,k},\tilde{Y}^{n,k},\tilde{Z}^{n,k}))_n$ be the subsequence of $((\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n))_n$ associated to $(M^{n,k}_T)_n$. Now, introduce the sequence of stopping times

$$\tau^k = \inf\left\{t\ge 0 : (\tilde{Y}^{n,k}_\cdot)^*_t\ge k,\ \text{for some } n\in\mathbb{N}\right\}\wedge T.$$

Let us show that (τ^k) is in fact a localizing sequence.

$$\begin{aligned}
P[\tau^k = T] &= P\left[(\tilde{Y}^{n,k}_\cdot)^*_T < k,\ \text{for all } n\in\mathbb{N}\right]\\
&\ge P\left[u^{-1}((M^{n,k}_\cdot - M_\cdot)^*_T + M^*_T) < k,\ \text{for all } n\in\mathbb{N}\right]\\
&= 1 - P\left[(M^{n,k}_\cdot - M_\cdot)^*_T + M^*_T\ge u(k),\ \text{for some } n\in\mathbb{N}\right]\\
&\ge 1 - P\left[\left\{(M^{n,k}_\cdot - M_\cdot)^*_T\ge 1,\ \text{for some } n\in\mathbb{N}\right\}\cup\left\{(M_\cdot)^*_T > u(k)-1\right\}\right]\\
&= 1 - P\left[(M^{n,k}_\cdot - M_\cdot)^*_T\ge 1,\ \text{for some } n\in\mathbb{N}\right] - P\left[(M_\cdot)^*_T > u(k)-1\right]\\
&\ge 1 - \sum_n P\left[(M^{n,k}_\cdot - M_\cdot)^*_T\ge 1\right] - P\left[(M_\cdot)^*_T > u(k)-1)\right]\\
&\ge 1 - \sum_n E\left[|M^{n,k}_T - M_T|\right] - \frac{E\left[(M_\cdot)^*_T\right]}{u(k)-1} \qquad (3.3)\\
&\ge 1 - \frac{1}{k} - \frac{E[(M_\cdot)^*_T]}{u(k)-1}\longrightarrow 1\\
&\qquad\qquad k\longrightarrow\infty,
\end{aligned}$$

where we used Markov's inequality to obtain (3.3). Therefore (τ^k) is a localizing sequence.

Let $n, k \in \mathbb{N}$, for all $t \in [0,T]$, $\tilde{Y}^{n,k}_{t\wedge\tau^k}$ is integrable. It follows from Jensen's inequality, since u^{-1} is convex, that for all $0 \le s \le t \le T$,

$$
\begin{aligned}
E\left[\tilde{Y}^{n,k}_{t\wedge\tau^k} \,|\, \mathcal{F}_s\right] &= E\left[u^{-1} \circ u(\tilde{Y}^{n,k}_{t\wedge\tau^k}) \,|\, \mathcal{F}_s\right] \\
&\ge u^{-1}\left(E\left[u(\tilde{Y}^{n,k}_{t\wedge\tau^k}) \,|\, \mathcal{F}_s\right]\right).
\end{aligned} \tag{3.4}
$$

On the set $\{\tau^k \le s\}$ it holds $E[u(\tilde{Y}^{n,k}_{\tau^k}) \,|\, \mathcal{F}_s] = u(\tilde{Y}^{n,k}_{\tau^k})$, and recalling that $u(\tilde{Y}^{n,k})$ is a submartingale, on the set $\{\tau_k > s\}$ it holds $E[u(\tilde{Y}^{n,k}_{t\wedge\tau^k}) \,|\, \mathcal{F}_s] \ge u(\tilde{Y}^{n,k}_{s\wedge\tau^k})$ by the optional sampling theorem. As u^{-1} is increasing, (3.4) leads to

$$
E\left[\tilde{Y}^{n,k}_{t\wedge\tau^k} \,|\, \mathcal{F}_s\right] \ge u^{-1}\left(u\left(\tilde{Y}^{n,k}_{s\wedge\tau^k}\right)\right) = \tilde{Y}^{n,k}_{s\wedge\tau^k}.
$$

Hence for all $n \in \mathbb{N}$, $\tilde{Y}^{n,k,\tau^k} := \tilde{Y}^{n,k}_{\cdot\wedge\tau^k}$ is a submartingale and $E[\tilde{Y}^{n,k}_{\tau^k} \,|\, \mathcal{F}_\cdot]$ is a martingale. By Doob-Meyer decomposition, see [27, Theorem 3.3.13], the càdlàg submartingale $\tilde{Y}^{n,k,\tau^k}$ admits the unique decomposition

$$
\tilde{Y}^{n,k}_{t\wedge\tau^k} = \tilde{Y}^{n,k}_0 + \tilde{A}^{n,k}_{t\wedge\tau^k} + \tilde{N}^{n,k}_{t\wedge\tau^k}, \quad t \in [0,T], \tag{3.5}
$$

where $\tilde{A}^{n,k,\tau^k}$ is an increasing predictable process starting at 0 and $\tilde{N}^{n,k,\tau^k}$ is a local martingale. Moreover, by Equation (2.2) and Lemma 3.3 there exists an increasing càdlàg process $\tilde{K}^{n,k}$ with $\tilde{K}^{n,k}_0 = 0$ such that

$$
\tilde{Y}^{n,k}_t = \tilde{Y}^{n,k}_0 + \int_0^t g_u(\tilde{Y}^{n,k}_u, \tilde{Z}^{n,k}_u)\,du + \tilde{K}^{n,k}_t - \int_0^t \tilde{Z}^{n,k}_u\,dW_u;
$$

where $\int g(\tilde{Y}^{n,k}, \tilde{Z}^{n,k})\,du + \tilde{K}^{n,k}$ is increasing, since g fulfills (POS), and is predictable. In addition $\int \tilde{Z}^{n,k}\,dW$ is a local martingale. By uniqueness of Doob-Meyer decomposition the processes $-\int \tilde{Z}^{n,k} 1_{[0,\tau^k]}\,dW$ and $\tilde{N}^{n,k,\tau^k}$ as well as $\int g(\tilde{Y}^{n,k}, \tilde{Z}^{n,k}) 1_{[0,\tau^k]}\,du + \tilde{K}^{n,k,\tau^k}$ and $\tilde{A}^{n,k,\tau^k}$ are indistinguishable. Then, from Equation (3.5) and $\tilde{Y}^{n,k}_t \ge 0$ we have for all $t \in [0,T]$

$$
\begin{aligned}
\int_0^{t\wedge\tau^k} \tilde{Z}^{n,k}_u\,dW_u &= \tilde{Y}^{n,k}_0 - \tilde{Y}^{n,k}_{t\wedge\tau^k} + \tilde{A}^{n,k}_{t\wedge\tau^k} \\
&\le V(x) + \tilde{A}^{n,k}_{\tau^k},
\end{aligned} \tag{3.6}
$$

where the last inequality comes from the fact that $(\tilde{Y}^{n,k}_0)_n$ increases to $V(x)$. On the other hand, since $(\tilde{Y}^{n,k}, \tilde{Z}^{n,k})$ satisfies (2.2) and g satisfies (POS),

$$
\begin{aligned}
\int_0^{t\wedge\tau^k} \tilde{Z}^{n,k}_u\,dW_u &\ge \tilde{Y}^{n,k}_0 - \tilde{Y}^{n,k}_{t\wedge\tau^k} + \int_0^{t\wedge\tau^k} g(\tilde{Y}^{n,k}_u, \tilde{Z}^{n,k}_u)\,du \\
&\ge -\tilde{Y}^{n,k}_{t\wedge\tau^k} \\
&\ge -E[\tilde{Y}^{n,k}_{\tau^k} \,|\, \mathcal{F}_{t\wedge\tau^k}],
\end{aligned} \tag{3.7}
$$

where the last inequality comes from the fact that $\tilde{Y}^{n,k,\tau^k}$ is a submartingale. Therefore, $\int \tilde{Z}^{n,k} 1_{[0,\tau^k]}\,dW$ is a supermartingale, as a local martingale bounded from below by the martingale $-E[\tilde{Y}^{n,k}_{\tau^k} \,|\, \mathcal{F}_{\cdot\wedge\tau^k}]$.

Hence, the inequalities (3.6) and (3.7) above lead to

$$\left| \int_0^{t\wedge\tau^k} \tilde{Z}_u^{n,k}\, dW_u \right| \le V(x) + \left| \tilde{Y}^{n,k}_{t\wedge\tau^k} \right| + \tilde{A}^{n,k}_{\tau^k},$$

which implies

$$\left(\int_0^{\cdot} \tilde{Z}_u^{n,k}\, dW_u \right)^*_{T\wedge\tau^k} \le V(x) + k + \tilde{A}^{n,k}_{\tau^k}.$$

The random variable $\tilde{A}^{n,k}_{\tau^k}$ is bounded in L^1, since we have $\tilde{A}^{n,k}_{\tau^k} = \tilde{Y}^{n,k}_0 - \tilde{Y}^{n,k}_{\tau^k} + \int_0^{\tau^k} \tilde{Z}^{n,k}_u\, dW_u$ with $(\tilde{Y}^{n,k}_0)_n$ increasing; $\int \tilde{Z}^{n,k} 1_{[0,\tau^k]}\, dW$ a P-supermartingale and $\tilde{Y}^{n,k}_{\tau^k}$ bounded. Hence, by Burkholder-Davis-Gundy's inequality, $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ is bounded in $\mathcal{L}^1$.

Step 4 Construction of the candidates $\bar{Z}$ and $\bar{Y}$. Now we are ready to construct the candidates maximizers for the control and the value processes. These constructions are based on compactness principles for the spaces $\mathcal{L}^1$ and L^1. Since $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ is $\mathcal{L}^1$ bounded, there exists, by means of [7, Theorem A], a sequence again denoted $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ in the asymptotic convex hull of $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ which converges in $\mathcal{L}^1$ along a localizing sequence $(\sigma^{n,k})_n$, and therefore $P \otimes dt$-a.s., to a process $\bar{Z}^k$. We obtain $\bar{Z}$ by implementing a diagonalization procedure such as in step 7 of the proof of [12, Theorem 4.1]: For another $k' > k$, we can find a subsequence $(\tilde{Z}^{n,k'})_n$ such that $(\tilde{Z}^{n,k'} 1_{[0,\tau^{k'}]} 1_{[0,\sigma^{n,k'}]})_n$ converges to a process $\bar{Z}^{k'}$ in $\mathcal{L}^1$ and $P \otimes dt$-a.s. By the same method, we can define the process $\bar{Z}$ by

$$\bar{Z}_0 = 0; \quad \bar{Z} = \sum_{k=1}^{\infty} \bar{Z}^k 1_{(\tau^{k-1},\tau^k]},$$

and put $\tilde{Z}^n = \tilde{Z}^{n,n}$ and $\sigma^{n,n} = \sigma^n$. Hence $(\tilde{Z}^n 1_{[0,\tau^n]} 1_{[0,\sigma^n]})$ converges to $\bar{Z}$ in $\mathcal{L}^1$ and $P \otimes dt$-a.s., but we also have $(\tilde{Z}^n 1_{[0,\tau^k]} 1_{[0,\sigma^k]})_n$ converges to $\bar{Z}^k$ for all k. Thus, by Burkholder-Davis-Gundy's inequality,

$$\int_0^{t\wedge\tau^k\wedge\sigma^k} \tilde{Z}^n_s\, dW_s \longrightarrow \int_0^{t\wedge\tau^k\wedge\sigma^k} \bar{Z}_s\, dW_s, \quad \text{for all } t,\ P\text{-a.s. and for each } k.$$

Taking the limit as $k \to \infty$ we have, for all t,

$$\int_0^t \tilde{Z}^n_u\, dW_u \longrightarrow \int_0^t \bar{Z}_u\, dW_u, \quad P\text{-a.s.} \tag{3.8}$$

Let $(\tilde{Y}^n)$ be a sequence in the asymptotic convex hull of (Y^n) corresponding to $(\tilde{Z}^n)$. For all $t \in [0,T]$ and $k \in \mathbb{N}$, we have $\tilde{Y}^n_{t\wedge\tau^k} = \tilde{Y}^n_0 + \tilde{A}^n_{t\wedge\tau^k} - \int_0^{t\wedge\tau^k} \tilde{Z}^n_u\, dW_u$. The sequence $(\tilde{A}^n_{T\wedge\tau^k})$ is bounded in L^1 as a consequence of the L^1-boundedness of $(A^n_{T\wedge\tau^k})_n$. Therefore, by Helly's theorem, we can find a subsequence in the asymptotic convex hull of $(\tilde{A}^{n,\tau^k})_n$ still denoted $(\tilde{A}^{n,\tau^k})_n$ such that, for k fixed, $(\tilde{A}^n_{t\wedge\tau^k})_n$ converges to $\tilde{A}_{t\wedge\tau^k}$ for all $t \in [0,T]$, P-a.s. and such that $\tilde{A}^{\tau^k}$ is an increasing positive integrable process with $\tilde{A}_0 = 0$. In particular, $(\tilde{A}^n_T)$ converges to $\tilde{A}_T$ P-a.s. Letting k go to infinity, $(\tilde{A}_{t\wedge\tau^k})_k$ converges to $\tilde{A}_t$, for all $t \in [0,T)$, P-a.s. Therefore we put

$$\tilde{Y}_t := \lim_{k\to\infty} \lim_{n\to\infty} \tilde{Y}^n_{t\wedge\tau^k} = V(x) + \tilde{A}_t - \int_0^t \hat{Z}_u\, dW_u; \quad t \in [0,T). \tag{3.9}$$

and for all $t \in [0, T)$, define

$$\bar{Y}_t := \lim_{s \downarrow t\, s \in \mathbb{Q}} \tilde{Y}_s = V(x) + \lim_{s \downarrow t\, s \in \mathbb{Q}} \tilde{A}_s - \int_0^t \hat{Z}_u \, dW_u$$

and $\bar{Y}_T := \tilde{Y}_T$. We claim that

$$\bar{Y} = \tilde{Y} \quad P \otimes dt\text{-a.s.} \tag{3.10}$$

This is because the jumps of $\tilde{Y}$ and $\bar{Y}$ coincide with the jumps of $\tilde{A}$, and being increasing, the latter process has countably many jumps.

Step 5 Construction of the candidate $\bar{X}$. Recall that since g satisfies (CONV), by Lemma 3.3 for all $n \in \mathbb{N}$ the triple $(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n)$, element of the asymptotic convex hull of $((X^n, Y^n, Z^n))_n$ is in $\mathcal{A}(x)$; and from *Step 1* we have $0 \le \tilde{X}^n_t \le \tilde{Y}^n_t$. Moreover, for each $n \in \mathbb{N}$ the process $\tilde{X}^n$ admits the representation

$$\tilde{X}^n_t = x + \int_0^t \tilde{\nu}^n_u \, dW^Q_u, \quad t \in [0, T]$$

for some predictable process $\tilde{\nu}^n \in \mathcal{L}^1(Q)$. Hence for all $t \in [0, T]$, for all $n \in \mathbb{N}$, we have

$$\left| \int_0^t \tilde{\nu}^n_u \, dW^Q_u \right| = \left| \tilde{X}^n_t - x \right| \le |\tilde{Y}^n_t| + x,$$

which implies, taking $(\tilde{\nu}^{n,k})_n$ to be the subsequence corresponding to $(M^{n,k})_n$, recall (3.2),

$$\left(\int_0^{\cdot} \tilde{\nu}^{n,k}_u \, dW^Q_u \right)^*_{T \wedge \tau^k} \le (\tilde{Y}^{n,k}_{\cdot})^*_{T \wedge \tau^k} + x \le k + x. \tag{3.11}$$

Therefore, by Burkholder-Davis-Gundy's inequality $(\tilde{\nu}^{n,k} 1_{[0,\tau^k]})_n$ is bounded in $\mathcal{L}^1(Q)$. With this local $\mathcal{L}^1(Q)$ bound at hand, we can use similar arguments as in *Step 4* to obtain a process $\bar{\nu}$ such that

$$\int_0^{t \wedge \tau^k} \tilde{\nu}^n_u \, dW^Q_u \longrightarrow \int_0^{t \wedge \tau^k} \bar{\nu}_u \, dW^Q_u \quad \text{for all } t,\ Q\text{-a.s. and for each k} \tag{3.12}$$

and

$$\int_0^t \tilde{\nu}^n_u \, dW^Q_u \longrightarrow \int_0^t \bar{\nu}_u \, dW^Q_u \quad \text{for all } t \in [0, T]\,,\ Q\text{-a.s.}$$

Put

$$\bar{X}_t = x + \int_0^t \bar{\nu}_u \, dW^Q_u. \tag{3.13}$$

Step 6 Verification. It follows from the definition of $\bar{Y}$ that $\bar{Y}_0 \ge V(x)$; let us verify that $(\bar{X}, \bar{Y}, \bar{Z})$ actually belongs to $\mathcal{A}(x)$. We start by showing that $\bar{X}$ is a wealth process. From $\tilde{X}^n \ge 0$ for all $n \in \mathbb{R}$, follows $\bar{X} \ge 0$. Since $\sigma\sigma'$ is of full rank, we can find a predictable process $\bar{\pi}$ such that $\bar{\pi}\sigma = \bar{\nu}$. Hence, from (3.11) and (3.12), $\bar{\pi}\sigma 1_{[0,\tau^k]} \in \mathcal{L}^1(Q)$ for all $k \in \mathbb{N}$ and therefore $\bar{\pi}\sigma \in \mathcal{L}(Q)$ and $d\bar{X}_t =$

$\bar{\pi}_u\sigma_u(\theta_u du + dW_u)$. Next let us show that $(\bar{Y}, \bar{Z}) \in \mathcal{A}^u(\xi + \bar{X}_T, g)$. To that end, we use an argument from [12] . By (3.10), there exists a set $B \subseteq \Omega \times [0,T]$ with $P \otimes dt(B^c) = 0$ such that $\bar{Y}_t(\omega) = \tilde{Y}_t(\omega)$ for all $(\omega, t) \in B$. Then, there exists a set $D \subseteq \{\omega : (\omega, t) \in B, \text{ for some } t\}$ with $P(D) = 1$ such that for all $\omega \in D$ the set $I(\omega) := \{t \in [0,T] : (\omega, t) \in B\}$ is a set of Lebesgue measure T and $\bar{Y}_t(\omega) = \tilde{Y}_t(\omega)$ for all $t \in I(\omega)$. Denote by λ_i^n, $n \le i \le \Lambda^n$, the convex weights of the convex combination $\tilde{Z}^n$. Let $s, t \in I$, $s \le t$, where I; s and t depend on $\omega \in D$. Using subsequently Fatou's lemma and (CONV) we are led to

$$
\begin{aligned}
&\bar{Y}_s + \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)\, du - \int_s^t \bar{Z}_u\, dW_u \\
&\le \lim_{k\to\infty} \liminf_{n\to\infty} \left(\tilde{Y}^n_{s\wedge\tau^k} + \int_{s\wedge\tau^k}^{t\wedge\tau^k} g_u(\tilde{Y}^n_u, \tilde{Z}^n_u 1_{[0,\sigma^n]}(u))\, du - \int_{s\wedge\tau^k}^{t\wedge\tau^k} \tilde{Z}^n_u\, dW_u \right) \\
&\le \lim_{k\to\infty} \liminf_{n\to\infty} \sum_{i=n}^{\Lambda^n} \lambda_i^n \left(Y^i_{s\wedge\tau^k} + \int_{s\wedge\tau^k}^{t\wedge\tau^k} g_u(Y^i_u, Z^i_u)\, du - \int_{s\wedge\tau^k}^{t\wedge\tau^k} Z^i_u\, dW_u \right) \\
&\le \lim_{k\to\infty} \liminf_{n\to\infty} \sum_{i=n}^{\Lambda^n} \lambda_i^n Y^i_{t\wedge\tau^k} = \lim_{k\to\infty} \liminf_{n\to\infty} \tilde{Y}^n_{t\wedge\tau^k} \\
&\qquad\qquad = \lim_{k\to\infty} \tilde{Y}_{t\wedge\tau^k} = \tilde{Y}_t = \bar{Y}_t.
\end{aligned} \tag{3.14}
$$

If s or t are not in I, then there exist two sequences (s_n) and (t_n) in I such that $s_n \downarrow s$, $t_n \downarrow t$ and $s_n \le t_n$. Equation (3.14) holds for each s_n, t_n. Namely,

$$
\bar{Y}_{s_n} + \int_{s_n}^{t_n} g(\bar{Y}_u, \bar{Z}_u)\, du - \int_{s_n}^{t_n} \bar{Z}_u\, dW_u \le \bar{Y}_{t_n}
$$

holds for all $n \in \mathbb{N}$. Since $\bar{Y}$ is right continuous and the integrals are continuous, taking the limit as n tends to infinity yields the desired result for s and t. Therefore, the pair $(\bar{Y}, \bar{Z})$ satisfies the inequality (2.2) with terminal condition $H = \xi + \bar{X}_T$ since for all $n \in \mathbb{N}$, $\tilde{Y}^n_T \le \xi + \tilde{X}^n_T$; and $(\tilde{Y}^n_T)$ and $(\tilde{X}^n_T)$ converges P-a.s. to $\bar{Y}_T$ and $\bar{X}_T$, respectively. Now let us show that $\bar{Y}$ is admissible and is a càdlàg process. Due to Lemmas 3.1 and 3.3 and positivity of u we have for all $n \in \mathbb{N}$ and $t \in [0,T]$

$$
\begin{aligned}
u\left(\tilde{Y}^n_t\right)^p &\le E\left[u\left(\xi + \tilde{X}^n_T\right) \mid \mathcal{F}_t\right]^p \\
&\le E\left[u(\xi + \tilde{X}^n_T)^p \mid \mathcal{F}_t\right],
\end{aligned}
$$

where we used Jensen's inequality. Taking expectation on both sides leads to $E[u(\tilde{Y}^n_t)^p] \le E[u(\xi + \tilde{X}^n_T)^p] \le C$. Hence, the family $(u(\tilde{Y}^n_t))_n$ is uniformly integrable, for all $t \in [0,T]$. Since for all n the process $\tilde{Y}^n$ is admissible, we have $u(\tilde{Y}^n_s) \le E[u(\tilde{Y}^n_t) \mid \mathcal{F}_s]$, $0 \le s \le t \le T$. Taking the limit as n goes to infinity, we obtain by means of continuity of u and dominated convergence theorem $u(\tilde{Y}_s) \le E[u(\tilde{Y}_t) \mid \mathcal{F}_s]$, i.e. $u(\tilde{Y})$ is a submartingale. The continuity property of the function u and definition of $\bar{Y}$ imply

$$
u(\bar{Y}_t) = \lim_{s\uparrow t, s\in\mathbb{Q}} u(\tilde{Y}_s),
$$

therefore by [22, Proposition 1.3.14], $u(\bar{Y})$ is a càdlàg submartingale, and $\bar{Y}$ is thus càdlàg as well. Hence $(\bar{X}, \bar{Y}, \bar{Z}) \in \mathcal{A}(x)$ and consequently $V(x) = \bar{Y}_0$, which ends the proof. □

Remarks 3.5. a) Unlike in [12] and [19] where minimal supersolutions of BSDEs are studied, we cannot guarantee that the stochastic integral of the process $\bar{Z}$ is a supermartingale even for a bounded terminal condition ξ. This is due to the fact that the random variable $\bar{X}_T$ may not be integrable.

b) In the above result, the assumption $\xi \in L^1_+(\Omega, \mathcal{F}_T, Q)$ can be replaced by $\xi \in L^2_+(\Omega, \mathcal{F}_T, P)$. This would cost a stronger integrability condition on the process θ. Indeed, if the martingale $\mathcal{E}(-\int \theta\, dW)$ satisfies the reverse Hölder inequality R_2 that is, there is a positive constance C such that for all stopping times $\tau \leq T$ it holds

$$E\left[\mathcal{E}\left(-\int \theta_u\, dW_u\right)_T^2 \mid \mathcal{F}_\tau\right]^{\frac{1}{2}} \leq C\mathcal{E}\left(-\int \theta_u\, dW_u\right)_\tau,$$

then by [10, Proposition 3] we have $E_Q[\xi] = E[\mathcal{E}(-\int_0^\cdot \theta_u\, dW_u)_T \xi] \leq CE[\xi^2]$ and therefore the first estimate of Lemma 3.1 remains valid. ◆

We finish this section with a direct consequence of Theorem 3.4 and its proof. Namely, existence of a maximal subsolution of a decoupled controlled FBSDE:

Corollary 3.6. *Assume that the generator g satisfies (*Conv*), (*Lsc*), (*Nor*) and (*Pos*); and $\xi \in L^\infty_+$. Then the system*

$$\begin{cases} Y_s & \leq Y_t - \int_s^t g(Y_u, Z_u)\, du + \int_s^t Z_u\, dW_u, \quad Y_T \leq \xi + X_T^\pi \\ X_t^\pi & = x + \int_0^t \pi_u \sigma_u \left(\theta_u\, du + dW_u\right), \quad \pi \in \Pi \end{cases} \tag{3.15}$$

admits a maximal subsolution. That is, there exists a control $\pi^* \in \Pi$ and a triple (X^{π^*}, Y^*, Z^*) satisfying (3.15) with $u\,(Y^*)$ being a submartingale such that for any control $\pi \in \Pi$ and any processes (X^π, Y, Z) satisfying (3.15) with $u(Y)$ a submartingale, we have $Y_0^* \geq Y_0$.

Proof. This follows from Theorem 3.4. □

3.2. Stability Results

In this section we assess the stability of maximal subsolutions with respect to the terminal condition and the generator. We will show that maximal subsolutions have a monotone stability with respect to both data. These stability results, already proved in [12] for minimal supersolution, will enable us to obtain a robust representation of the operator $\mathcal{E}_0^g$.

Proposition 3.7. *Assume that the generator g satisfies (*Conv*), (*Lsc*), (*Nor*) and (*Pos*). Let $(\xi^n) \subseteq L^\infty_+$. If (ξ^n) decreases pointwise to a random variable ξ, then $\mathcal{E}_0^g(\xi) = \lim_{n\to\infty} \mathcal{E}_0^g(\xi^n)$.*

Proof. See Appendix A. □

Proposition 3.8. *Let $\xi \in L^\infty_+$ be a terminal condition, and (g^n) be a sequence of generators decreasing pointwise to g. Assume that each function satisfies (*Conv*), (*Lsc*), (*Nor*) and (*Pos*). Then $\mathcal{E}_0^g(\xi) = \lim_{n\to\infty} \mathcal{E}_0^{g^n}(\xi)$.*

Proof. See Appendix A. □

4. Representation and Characterization

In the previous section we obtained existence of optimal trading strategies of our control problem. This was a rather abstract result, and only gave us little information on how one could compute such an optimizer or how it depends on the other parameters. The point of this section is to find a characterization of the optimal controls of Problem (2.3).

4.1. Robust Representation

We consider the set

$$\mathcal{D} := \left\{ \beta : \beta \text{ predictable and } \int_0^T |\beta_u| \, du < \infty \right\}.$$

For any $\beta \in \mathcal{D}$ and $q \in \mathcal{L}$, we define, for $0 \leq s \leq t \leq T$

$$\frac{dQ^q}{dP} = \exp\left(\int_0^T q_u \, dW_u - \frac{1}{2} \int_0^T \|q_u\|^2 \, du \right) \quad \text{and} \quad D^\beta_{s,t} := e^{-\int_s^t \beta_u \, du}, \quad t \in [0,T].$$

We also define the set

$$\mathcal{Q} := \left\{ q \in \mathcal{L} : \frac{dQ^q}{dP} \in L^1_+ \right\}.$$

For any admissible trading strategy $\pi \in \Pi$, the associated wealth process is given by $dX^\pi_t = \pi_t \sigma_t (\theta_t \, dt + dW_t)$, with $X^\pi_0 = x$ and $X^\pi \geq 0$. Let $0 \leq s \leq t \leq T$, and consider the functional

$$\mathcal{E}^g_{s,t}(H) := \operatorname{ess\,sup} \left\{ Y_s : (Y,Z) \in \mathcal{A}^u(H,g) \right\}, \quad H \in L^0(\mathcal{F}_t).$$

Recall that $\mathcal{A}^u(H,g)$ is the set of subsolutions $(Y,Z) \in \mathcal{S}_+ \times \mathcal{L}$ of the BSDE with terminal condition H and generator g such that $u(Y)$ is a submartingale. In particular, $\mathcal{E}^g_0(H) = \mathcal{E}^g_{0,T}(H)$ for all $H \in L^0(\mathcal{F}_T)$. Let $\tau \leq \gamma$ be two stopping times valued in $[0,T]$. For any $\pi \in \Pi$, define

$$\Theta_{\tau,\gamma}(\pi) := \left\{ \pi' \in \Pi : \pi' 1_{[\tau,\gamma]} = \pi 1_{[\tau,\gamma]} \right\}$$

and

$$Y_\tau(X^\pi_\tau) := \operatorname*{ess\,sup}_{\pi' \in \Theta_{0,\tau}(\pi)} \mathcal{E}^g_{\tau,T}\left(X^{\pi'}_T + \xi \right), \tag{4.1}$$

where $\xi \in L^\infty_+$ is the random endowment. We define the convex conjugate g^* of the generator g by

$$g^*(\beta, q) := \sup_{y \in \mathbb{R}_+, z \in \mathbb{R}^d} \left\{ \beta y + qz - g(y,z) \right\}, \quad \beta \in \mathbb{R}, q \in \mathbb{R}^d.$$

Consider the condition

(ADM) $g(y,z) \geq -1/2 \|z\|^2 u''(y)/u'(y)$ on $\mathbb{R}_+ \times \mathbb{R}^d$.

The following theorem gives a robust representation of $\mathcal{E}^g_{0,\tau}$.

Theorem 4.1. *Assume that the generator g satisfies (CONV), (LSC), (NOR), (POS) and (ADM). Then, for every $\pi \in \Pi$ and any stopping time $0 \leq \tau \leq T$, the following robust representation holds:*

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = \inf_{(\beta,q) \in \mathcal{D} \times \mathcal{Q}} E_{Q^q}\left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u) \, du \right], \quad \pi \in \Pi. \tag{4.2}$$

For the proof of the theorem we need the following lemma.

Lemma 4.2. *Assume $H \in L^\infty$. Let f be a function satisfying (ADM) and such that the BSDE with terminal condition H and generator f has a solution $(Y,Z) \in \mathcal{S} \times \mathcal{L}^1$ satisfying $Y \geq c$ for some $c > 0$. Then, $u(Y)$ is a submartingale.*

Proof. By Itô's formula it holds

$$u(Y_t) = u(Y_0) + \int_0^t \left(u'(Y_u) f(Y_u, Z_u) + \frac{1}{2} u''(Y_u) Z_u^2 \right) du - \int_0^t u'(Y_u) Z_u \, dW_u, \tag{4.3}$$

for all $t \in [0, T]$. Therefore since $Y > 0$, due to (ADM) we have $u'(Y_u) f(Y_u, Z_u) + \frac{1}{2} u''(Y_u) Z_u^2 \geq 0$ so that the second term of the right hand side in (4.3) defines an increasing process. Thus, as $H \in L^\infty$ and $Y \geq c$, $u(Y)$ is a submartingale. In other words, (Y, Z) is an admissible subsolution of the BSDE with terminal condition H and generator f. $\square$

Proof (proof of Theorem 4.1). Let $\tau \leq T$ be a stopping time. For every $\pi \in \Pi$ and $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$, if $\mathcal{A}^u(Y_\tau(X_\tau^\pi), g) \neq \emptyset$, let $(Y, Z) \in \mathcal{A}^u(Y_\tau(X_\tau^\pi), g)$. There exists a càdlàg increasing process K with $K_0 = 0$ such that on $\{t \leq \tau\}$,

$$Y_t = Y_s + \int_s^t g_u(Y_u, Z_u) \, du - \int_s^t Z_u \, dW_u + K_t - K_s, \quad 0 \leq s \leq t.$$

Define the localizing sequence of stopping times (σ_n) by

$$\sigma_n := \inf \left\{ t \geq 0 : \left| \int_0^t D_{0,u}^\beta Z_u \, dW_u \right| \geq n \right\} \wedge T.$$

Applying Itô's formula to $D_{0,t}^\beta Y_t$ and Girsanov's theorem such as in [16], we have

$$Y_0 \leq E_{Q^q} \left[D_{0,t\wedge\sigma_n}^\beta Y_{t\wedge\sigma_n} + \int_0^{t\wedge\sigma_n} D_{0,u}^\beta g_u^*(\beta_u, q_u) \, du \right], \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, T].$$

Since g satisfies (NOR) the function g^* is positive. Using the fact that (σ_n) is a localizing sequence, there is n large enough such that $\tau \leq \sigma_n$; and since $Y_\tau \leq Y_\tau(X_\tau^\pi)$ and D^β is positive, we have

$$Y_0 \leq E_{Q^q} \left[D_{0,\tau}^\beta Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^\beta g_u^*(\beta_u, q_u) \, du \right].$$

Therefore,

$$\mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^\pi)) \leq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q} \left[D_{0,\tau}^\beta Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^\beta g_u^*(\beta_u, q_u) \, du \right]. \tag{4.4}$$

If $\mathcal{A}^u(Y_\tau(X_\tau^\pi), g) = \emptyset$, (4.4) is obvious.

On the other hand, for each $k \in \mathbb{N}$ and $\pi \in \Pi$ we define $H^k(\pi) := Y_\tau(X_\tau^\pi) \wedge k$, which is a bounded $\mathcal{F}_\tau$-random variable. Defining for every $n \in \mathbb{N}$ the function g^n on $\mathbb{R}_+ \times \mathbb{R}^d$ by

$$g^n(y, z) := \sup_{|\beta|\leq n; \|q\|\leq n} \{\beta y + qz - g^*(\beta, q)\} \vee -\frac{1}{2} u''(y) \|z\|^2 / u'(y),$$

the sequence (g^n) converges pointwise to g as a consequence of the Fenchel-Moreau theorem. In addition, for each $n \in \mathbb{N}$ the function g^n satisfies the quadratic growth condition

$$g^n(y, z) \leq C_n(1 + |y| + \|z\|^2), \quad y \in \mathbb{R}, \, z \in \mathbb{R}^d, \; C_n \geq 0.$$

Fixing $n \in \mathbb{N}$, for every $k \in \mathbb{N}$ there exists $(Y^{n,k}, Z^{n,k}) \in \mathcal{S} \times \mathcal{L}^1$ solution of the BSDE with terminal condition $H^k(\pi)$ and driver g^n, see for instance [5]. It follows from [16] that there exist predictable processes (β^n, q^n) satisfying $|\beta^n| \leq C_n$ and $\int q^n \, dW \in BMO$ such that on $\{t \leq \tau\}$

$$Y_t^{n,k} = E_{Q^{q^n}}\left[D_{t,\tau}^{\beta^n} H^k(\pi) + \int_t^\tau D_{0,u}^{\beta^n} g_u^{n,*}(\beta_u^n, q_u^n)\, du \,\Big|\, \mathcal{F}_t \right], \quad P\text{-a.s.}, \tag{4.5}$$

where $g^{n,*}$ is the convex conjugate of g^n. In particular, since g satisfies (NOR), we have $\beta y - g^*(\beta, q) \leq 0$ for all β, q so that g^n also satisfies (NOR). Thus, it holds $g^{n,*} \geq 0$, and from $(x, x, 0) \in \mathcal{A}(x)$ it follows $H^k(\pi) \geq x$, which yields $Y_t^{n,k} \geq E_{Q^{q^n}}[D_{t,\tau}^{\beta^n} x] > 0$. Since $g^n(y,z) \geq -\frac{1}{2} u''(y) \|z\|^2 / u'(y)$, it follows from Lemma 4.2 that $u(Y^{n,k})$ is a submartingale. That is, $(Y^{n,k}, Z^{n,k})$ is an admissible subsolution of the BSDE with generator g^n and terminal condition $H^k(\pi)$. Therefore, $\mathcal{E}_0^{g^n}(H^k(\pi)) \geq Y_0^{n,k}$. Taking the limit as k goes to infinity, it follows from the monotone stability of Proposition 3.7 and the monotone convergence theorem that

$$\mathcal{E}_{0,\tau}^{g^n}(Y_\tau(X_\tau^\pi)) \geq E_{Q^{q^n}}\left[D_{0,\tau}^{\beta^n} Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^{\beta^n} g_u^{n,*}(\beta_u^n, q_u^n)\, du \right] \quad \text{for all } n \in \mathbb{N}.$$

Since $(\beta^n, q^n) \in \mathcal{D} \times \mathcal{Q}$ for each n, we have

$$\mathcal{E}_{0,\tau}^{g^n}(Y_\tau(X_\tau^\pi)) \geq \inf_{(\beta,q) \in \mathcal{D} \times \mathcal{Q}} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^{\beta} g_u^{n,*}(\beta_u, q_u)\, du \right].$$

Using $g^* \leq g^{n,*}$ for all $n \in \mathbb{N}$ and then taking the limit as n goes to infinity, the monotone stability of Proposition 3.8 yields the second inequality, which concludes the proof. □

Proposition 4.3. *Under the assumptions of Theorem 4.1, for any $[0,T]$-valued stopping time τ, it holds*

$$V(x) = \sup_{\pi \in \Pi} \inf_{(\beta,q) \in \mathcal{D} \times \mathcal{Q}} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^{\beta} g_u^*(\beta_u, q_u)\, du \right]. \tag{4.6}$$

Proof. We have

$$V(x) = \sup_{\pi \in \Pi} \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^\pi)). \tag{4.7}$$

In fact,

$$\begin{aligned} \sup_{\pi \in \Pi} \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^\pi)) &= \sup_{\pi \in \Pi} \mathcal{E}_{0,\tau}^g\left(\operatorname*{ess\,sup}_{\pi' \in \Theta_{0,\tau}(\pi)} \mathcal{E}_{\tau,T}^g\left(X_T^{\pi'} + \xi \right) \right) \\ &= \sup_{\pi \in \Pi} \sup_{\pi' \in \Theta_{0,\tau}(\pi)} \mathcal{E}_{0,T}^g\left(X_T^{\pi'} + \xi \right) = V(x), \end{aligned}$$

where we used monotonicity and flow property of the operators $\mathcal{E}_{s,t}^g(\cdot)$, $0 \leq s \leq t \leq T$, see [12, Proposition 3.6]. By Theorem 4.1 the proof is done. □

4.2. Existence of a Saddle Point

Considering the dual representation of $\mathcal{E}^g_{0,\tau}$ derived in Theorem 4.1, a pair $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$ is said to be a subgradient of $\mathcal{E}^g_{0,\tau}$ at $Y_\tau(X^\pi_\tau)$ if

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = E_{Q^q}\left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u)\, du\right].$$

In the case where the generator only depends on z, equivalence between existence of a subgradient of a monetary utility function and quadratic growth of the driver g was proved by Delbaen et al. [9]. The following result uses their compactness argument. We will also need the conditions

(QG) quadratic growth: $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and $\forall \eta > 0$ there exists $C > 0$: $g(y, z) \leq C(1 + |y| + \|z\|^2)$ for all $y \in \mathbb{R}$: $|y| \geq \eta$ and $z \in \mathbb{R}^d$.

Theorem 4.4. *Assume that g satisfies (ADM), (CONV), (LSC), (QG), (NOR) and (POS). Then, $\mathcal{E}^g_0$ admits a local subgradient: For any $[0, T]$-valued stopping time τ and any $\pi \in \Pi$, $\mathcal{E}^g_{0,\tau}$ admits a subgradient $(q^\tau, \beta^\tau) \in \mathcal{D} \times \mathcal{Q}$ at $Y_\tau(X^\pi_\tau)$.*

Proof. Let $\pi \in \Pi$ be fixed for the rest of the proof. Let $\eta > 0$ in (QG). Due to Theorem 4.1, we have

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = \inf_{\frac{dQ^q}{dP} D^\beta_{0,\tau} \in \mathcal{K}} \left\{ E_{Q^q}\left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u)\, du\right]\right\},$$

where

$$\mathcal{K} := \left\{\frac{dQ^q}{dP} D^\beta_{0,\tau} : (\beta, q) \in \mathcal{D} \times \mathcal{Q}\right\} \subseteq L^1.$$

For every $k \geq 0$ the set

$$\Gamma_\tau := \left\{\frac{dQ^q}{dP} D^\beta_{0,\tau} \in \mathcal{K} : E_{Q^q}\left[\int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u)\, du\right] \leq k\right\} \tag{4.8}$$

is convex, see [13]. Let us show that it is $\sigma(L^1, L^\infty)$-compact. Let $(\beta, q) \in \mathbb{R} \times \mathbb{R}^d$ be given. By definition, we have

$$\begin{aligned} g^*(\beta, q) &= \sup_{y \in \mathbb{R}, z \in \mathbb{R}^d} \{\beta y + qz - g(y, z)\} \\ &\geq \sup_{|y| \geq \eta, z \in \mathbb{R}^d} \{\beta y + qz - g(y, z)\} \\ &\geq \sup_{|y| \geq \eta, z \in \mathbb{R}^d} \left\{\beta y + qz - C(1 + y + \|z\|^2)\right\} \\ &\geq \sup_{|y| \geq \eta} \{\beta y - Cy\} + b\|q\|^2 - C, \end{aligned}$$

with $b = \frac{1}{4C}$. If $|\beta| > C$, then let $n \in \mathbb{N}$ be big enough such that $y := n\beta$ satisfies $|y| \geq \eta$. Then,

$$\sup_{|y| \geq \eta} \{-\beta y - C|y|\} \geq n|\beta|(|\beta| - C),$$

so that $g^*(\beta, q) = \infty$. Therefore, we can restrict ourselves to $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$ with $|\beta| \leq C$. Hence, we can find a positive constant a such that

$$g^*(\beta, q) \geq a\beta + b\|q\|^2 - C. \tag{4.9}$$

Since β is bounded, $D^{\beta}_{0,u} = e^{-\int_0^u \beta_r \, dr}$ is bounded as well. Thus multiplying both sides of (4.9) by $D^{\beta}_{0,u}$ and integrating with respect to $Q^q \otimes dt$ lead to

$$E_{Q^q}\left[\int\limits_0^\tau D^{\beta}_{0,u} g^*(\beta_u, q_u)\, du\right] \geq A_1 + A_2 E_{Q^q}\left[\int\limits_0^\tau \|q_u\|^2 \, du\right],$$

where A_1 and A_2 are positive constants which do not depend on β and q. Arguing similar to the proof of [9, Theorem 2.2], we can find a positive constant c such that

$$\left\{\frac{dQ^q}{dP} D^{\beta}_{0,\tau} \in \mathcal{K} : E_{Q^q}\left[\int\limits_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right] \leq k\right\}$$
$$\subseteq \left\{\frac{dQ^q}{dP} D^{\beta}_{0,\tau} \in \mathcal{K} : E\left[\frac{dQ^q}{dP} \log \frac{dQ^q}{dP}\right] \leq c\right\}$$

and therefore, we can conclude using the de la Vallée Poussin theorem that the left hand side in the above inclusion is L^1- uniformly integrable. We take a maximizing sequence $(\frac{dQ^{q^n}}{dP} D^{\beta^n}_{0,\tau})_n$ for the functional $\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau))$. Since $Y_\tau(X^\pi_\tau)$ is positive, it follows that the sequence $(E_{Q^{q^n}}[\int_0^\tau D^{\beta^n}_{0,u} g^*_u(\beta^n_u, q^n_u)\, du])_n$ admits a subsequence which is bounded from above. Therefore, the previous step shows that the sequence $(\frac{dQ^{q^n}}{dP} D^{\beta^n}_{0,\tau})_n$ is uniformly integrable. In addition, applying a compactness argument of Komlos type, we can find a sequence denoted $(\tilde{M}^n_T)$ in the asymptotic convex hull of $(\frac{dQ^{q^n}}{dP} D^{\beta^n}_{0,\tau})_n$ which converges P-a.s. to the limit $M_T \in L^0_+$. The sequence $(\tilde{M}^n_T)$ is as well uniformly integrable and therefore converges to M_T in L^1. By the arguments used in the proof of [13, Theorem 3.10], it is possible to show that for all $n \in \mathbb{N}$ there exist $\tilde{q}^n$ and $\tilde{\beta}^n$ such that $\tilde{M}^n_T = \frac{dQ^{\tilde{q}^n}}{dP} D^{\tilde{\beta}^n}_{0,\tau}$ and, up to other convex combinations, the sequences $(\tilde{q}^n)$ and $(\tilde{\beta}^n)$ converge $P \otimes dt$-a.s. to some q^τ and β^τ, respectively and $M_T = \frac{dQ^{q^\tau}}{dP} D^{\beta^\tau}_{0,\tau}$. Since $|\tilde{\beta}^n| \leq C$ for all n, it holds $|\beta^\tau| \leq C$. By Fatou's lemma and convexity, we have

$$\begin{aligned}\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) &= \liminf_{n\to\infty} E_{Q^{q^n}}\left[D^{\beta^n}_{0,\tau} Y_\tau(X^\pi_\tau) + \int\limits_0^\tau D^{\beta^n}_{0,u} g^*(\beta^n_u, q^n_u)\, du\right] \\ &\geq E\left[\liminf_{n\to\infty} \frac{dQ^{\tilde{q}^n}}{dP}\left(D^{\tilde{\beta}^n}_{0,\tau} Y_\tau(X^\pi_\tau) + \int\limits_0^\tau D^{\tilde{\beta}^n}_{0,u} g^*(\tilde{\beta}^n_u, \tilde{q}^n_u)\, du\right)\right].\end{aligned}$$

Lower-semicontinuity of g^* yields

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) \geq E_{Q^{q^\tau}}\left[D^{\beta^\tau}_{0,\tau} Y_\tau(X^\pi_\tau) + \int\limits_0^\tau D^{\beta^\tau}_{0,u} g^*(\beta^\tau_u, q^\tau_u)\, du\right].$$

Since $|\beta^\tau| \leq C$ and $M_T \in L^1$, we have $\beta^\tau \in \mathcal{D}$ and $q^\tau \in \mathcal{Q}$. $\square$

Corollary 4.5. *Under the assumptions of Theorem 4.4, for any optimal strategy $\pi^* \in \Pi$ and any $[0,T]$-valued stopping time τ one has*

$$V(x) = \mathcal{E}^g_{0,\tau}\left(Y_\tau(X^{\pi^*}_\tau)\right). \tag{4.10}$$

In addition, Problem (2.3) admits a local saddle point in the sense that, there exists $(\beta^\tau, q^\tau) \in \mathcal{D} \times \mathcal{Q}$ satisfying

$$\begin{aligned} V(x) &= E_{Q^{q^\tau}} \left[D^{\beta^\tau}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*_u(\beta^\tau_u, q^\tau_u)\, du \right] \\ &= \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q} \left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du \right]. \end{aligned}$$

Proof. By definition of $Y_\tau(X^{\pi^*}_\tau)$, monotonicity and the flow property of $\mathcal{E}^g_{s,t};\, 0 \le s \le t \le T$, we have

$$\begin{aligned} \mathcal{E}^g_{0,\tau}(Y_\tau(X^{\pi^*}_\tau)) &= \mathcal{E}^g_{0,\tau} \left(\operatorname*{ess\,sup}_{\pi\in\Theta_{0,\tau}(\pi^*)} \mathcal{E}^g_{\tau,T}\left(X^\pi_T + \xi\right) \right) \\ &= \sup_{\pi\in\Theta_{0,\tau}(\pi^*)} \mathcal{E}^g_{0,T}\left(X^\pi_T + \xi\right) \ge V(x), \end{aligned}$$

since $\pi^* \in \Theta_{0,\tau}(\pi^*)$. Thus, Equation (4.10) is a consequence of Equation (4.7).

It follows from Theorem 4.4 and Equation (4.10) that there exists $(\beta^\tau, q^\tau) \in \mathcal{D} \times \mathcal{Q}$ such that

$$V(x) = E_{Q^{q^\tau}} \left[D^{\beta^\tau}_{0,T} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*_u(\beta^\tau_u, q^\tau_u)\, du \right] \tag{4.11}$$

and for every $\pi \in \Pi$ exists $(\beta(\pi), q(\pi)) \in \mathcal{D} \times \mathcal{Q}$ such that

$$\mathcal{E}^g_{0,\tau}\left(Y_\tau(X^\pi_\tau)\right) = E_{Q^{q(\pi)}} \left[D^{\beta(\pi)}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta(\pi)}_{0,u} g^*_u(\beta_u(\pi), q_u(\pi))\, du \right].$$

Thus, taking the supremum with respect to π on both sides yields

$$\begin{aligned} \mathcal{E}^g_{0,\tau}\left(Y_\tau(X^{\pi^*}_\tau)\right) &= \sup_{\pi\in\Pi} E_{Q^{q(\pi)}} \left[D^{\beta(\pi)}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta(\pi)}_{0,u} g^*_u(\beta_u(\pi), q_u(\pi))\, du \right] \\ &\ge \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q} \left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du \right]. \end{aligned}$$

Since we always have $\inf\sup \ge \sup\inf$, it follows that

$$\begin{aligned} &\sup_{\pi\in\Pi} \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q} \left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du \right] \\ &\quad = E_{Q^{q^\tau}} \left[D^{\beta^\tau}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*_u(\beta^\tau_u, q^\tau_u)\, du \right] \\ &\quad = \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q} \left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du \right]. \end{aligned}$$

The proof is complete. □

Remark 4.6. If g defined on the space $\mathbb{R}\times\mathbb{R}^d$ satisfies (ADM), (CONV), (NOR), (POS) and (QG), then one can take $\tau = T$ in Equation (4.1), that is $Y_\tau(X^\pi_\tau) = X^\pi_T + \xi$, and work on the whole time interval $[0,T]$ in the proof of Theorem 4.4 and the subsequent corollary. The main reason for working with stopping times is to allow for generators that satisfy the conditions (CONV), (NOR) and (POS) only on a subset $I \times \mathbb{R}^d$, where $I \subseteq \mathbb{R}_+$ is an open interval as in the following example. ♦

Example 4.7 (Certainty equivalent). Let us come back to the certainty equivalent example of Section 2. For $u(x) = \log(x)$, Equation (2.5) becomes

$$Y_t = X - \frac{1}{2}\int_t^T \frac{|Z_u|^2}{Y_u}\,du + \int_t^T Z_u\,dW_u, \quad t \in [0,T].$$

The generator $g(y,z) = \frac{1}{2}|z|^2/y$ satisfies (LSC), (CONV), (NOR) and (POS) on $(0,\infty)\times\mathbb{R}^d$ and it can be extended on $\mathbb{R}_+\times\mathbb{R}^d$ to a generator satisfying the same conditions by putting

$$g(y,z) = \begin{cases} \frac{1}{2}\frac{|z|^2}{y} & \text{if } y > 0 \\ 0 & \text{if } z = 0 \\ +\infty & \text{if } y = 0, z \neq 0. \end{cases}$$

Hence, Theorem 3.4 ensures the existence of an optimal trading strategy $\pi^* \in \Pi$. However, if we consider the function on $\mathbb{R}_+\times\mathbb{R}^d$, we can not guarantee, with our method, that the set Γ_τ defined in (4.8) is weakly compact and therefore that the problem admits a saddle point. A way around is to introduce a stopping time $0 < \tau \le T$ and work locally on $[0,\tau]$ as follows: Let $\pi^* \in \Pi$ be an optimal strategy and put $Y^\pi_t := u^{-1}(E[u(X^\pi_T + \xi)\,|\,\mathcal{F}_t])$. Since $x > 0$, there exists $m \in \mathbb{N}$ such that $x \ge \frac{1}{m}$. Define the stopping time τ by

$$\tau := \inf\left\{t \ge 0 : X^{\pi^*}_t \le \frac{1}{m}\right\} \wedge T.$$

We can restrict the study to subsolutions $(Y,Z) \in \mathcal{A}^u(X^\pi_T + \xi)$ satisfying $Y \ge X^\pi$, for all $t \in [0,T]$. Hence, with BSDE duality we have $Y^{\pi^*}_{\tau\wedge t} \ge X^{\pi^*}_{\tau\wedge t} \ge \frac{1}{m}$. Applying martingale representation theorem and Itô's formula such as in Example 2.1, we can find a process $Z^{\pi^*} \in \mathcal{L}^1$ such that

$$Y^{\pi^*}_t = Y^{\pi^*}_\tau - \int_t^\tau g_u(Y^{\pi^*}_u, Z^{\pi^*}_u)\,du + \int_t^\tau Z^{\pi^*}_u\,dW_u \quad \text{on } \{t \le \tau\}.$$

Since the set $\{Y_\tau : (X,Y,Z) \in \mathcal{A}(x)\}$ is upward directed, using the arguments of Theorem 3.4 we can find a strategy $\bar\pi \in \Pi$ such that

$$Y_\tau(X^{\pi^*}_\tau) := \operatorname*{ess\,sup}_{\pi' \in \Theta_{0,\tau}(\pi^*)} \mathcal{E}_{\tau,T}(X^{\pi'}_T + \xi) = Y^{\bar\pi}_\tau = \mathcal{E}^g_{\tau,T}(X^{\bar\pi}_T + \xi)$$

with $\bar\pi \in \Theta_{0,\tau}(\pi^*)$, i.e. $\bar\pi 1_{[0,\tau]} = \pi^* 1_{[0,\tau]}$ and $\bar\pi \in \Pi$. Moreover, since $\Theta_{0,\tau}(\pi^*) = \Theta_{0,\tau}(\bar\pi)$, we have $Y_\tau(X^{\pi^*}_\tau) = Y_\tau(X^{\bar\pi}_\tau)$. By $Y^{\bar\pi}_{t\wedge\tau} \ge X^{\bar\pi}_{t\wedge\tau} = X^{\pi^*}_{t\wedge\tau} \ge \frac{1}{m} > 0$, we also get

$$\begin{aligned} Y^{\bar\pi}_0 &= Y^{\bar\pi}_\tau - \int_0^\tau g_u(Y^{\bar\pi}_u, Z^{\bar\pi}_u)\,du + \int_0^\tau Z^{\bar\pi}_u\,dW_u \\ &= Y_\tau(X^{\bar\pi}_\tau) - \int_0^\tau g_u(Y^{\bar\pi}_u, Z^{\bar\pi}_u)\,du + \int_0^\tau Z^{\bar\pi}_u\,dW_u. \end{aligned}$$

For almost every (ω, t) such that $t \leq \tau(\omega)$ the function g is differentiable at $(Y_t^{\bar{\pi}}(\omega), Z_t^{\bar{\pi}}(\omega))$ and it admits a unique subgradient $(\bar{\beta}_t(\omega), \bar{q}_t(\omega))$ given by

$$\bar{q}_t = \frac{Z_t^{\bar{\pi}}}{Y_t^{\bar{\pi}}} \quad \text{and} \quad \bar{\beta}_t = -\frac{|Z_t^{\bar{\pi}}|^2}{2(Y_t^{\bar{\pi}})^2} \quad \text{on } \{t \leq \tau\}.$$

Since $Y_{t\wedge\tau}^{\bar{\pi}} \geqq 1/m$ and $Z^{\bar{\pi}} \in \mathcal{L}^1$, it follows that $(\bar{\beta}, \bar{q}) \in \mathcal{D} \times \mathcal{Q}$ and we have $g_t(Y_t^{\bar{\pi}}, Z_t^{\bar{\pi}}) = \bar{\beta}_t Y_t^{\bar{\pi}} + \bar{q}_t Z_t^{\bar{\pi}} - g_t^*(\bar{\beta}_t, \bar{q}_t)$. Thus, using the arguments leading to Equation 4.5, one has

$$Y_0^{\bar{\pi}} = E_{Q^{\bar{q}}}\left[D_{0,\tau}^{\bar{\beta}} Y_\tau(X_\tau^{\bar{\pi}}) + \int_0^\tau g_u^*(\bar{\beta}_u, \bar{q}_u)\,du\right]. \tag{4.12}$$

But since for every $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$ it holds

$$Y_0^{\bar{\pi}} \leq E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\bar{\pi}}) + \int_0^\tau g_u^*(\beta_u, q_u)\,du\right],$$

it follows,

$$\begin{aligned} Y_0^{\bar{\pi}} &= \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\bar{\pi}}) + \int_0^\tau g_u^*(\beta_u, q_u)\,du\right] \\ &= \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\bar{\pi}})) \end{aligned} \tag{4.13}$$

where the second equality above follows from the representation theorem 4.1. By the identity $Y_\tau(X_\tau^{\pi^*}) = Y_\tau(X_\tau^{\bar{\pi}})$, one has $\mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\bar{\pi}})) = \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\pi^*}))$, so that it follows from the equations (4.12) and (4.13) that $\mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\pi^*}))$ admits the subgradient $(\bar{\beta}, \bar{q})$. Therefore, the utility maximization problem $V(x) = \sup_{\pi\in\Pi} C_0(X_T^\pi + \xi)$ can be written as a robust control problem admitting a local saddle point in the sense of Corollary 4.5. In fact,

$$\begin{aligned} &\inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\,du\right] \\ &\leq \sup_{\pi\in\Pi} E_{Q^{\bar{q}}}\left[D_{0,\tau}^{\bar{\beta}} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\bar{\beta}_u, \bar{q}_u)\,du\right] \\ &\leq \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\pi^*})) = E_{Q^{\bar{q}}}\left[D_{0,\tau}^{\bar{\beta}} Y_\tau(X_\tau^{\pi^*}) + \int_0^\tau g_u^*(\bar{\beta}_u, \bar{q}_u)\,du\right] \\ &\leq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi^*}) + \int_0^\tau g_u^*(\beta_u, q_u)\,du\right] \\ &\leq \sup_{\pi\in\Pi} \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\,du\right] \\ &\leq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\,du\right]. \end{aligned}$$

To justify the second inequality above, notice that with the arguments leading to (4.4), we have

$$\mathcal{E}^g_{\tau,T}(X^\pi_T+\xi) \le E_{Q^{\bar q}}\left[D^{\bar\beta}_{\tau,T}(X^\pi_T+\xi) + \int_\tau^T g^*_u(\bar\beta_u,\bar q_u)\,du \,|\, \mathcal{F}_\tau\right].$$

Therefore,

$$\begin{aligned}
&\sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&= \sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} \operatorname*{ess\,sup}_{\pi'\in\Theta_{0,\tau}(\pi)} \mathcal{E}^g_{\tau,T}(X^{\pi'}_T+\xi) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&= \sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau}\mathcal{E}^g_{\tau,T}(X^{\pi}_T+\xi) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau}E_{Q^{\bar q}}\left[D^{\bar\beta}_{\tau,T}(X^\pi_T+\xi) + \int_\tau^T g^*_u(\bar\beta_u,\bar q_u)\,du \,|\, \mathcal{F}_\tau\right] + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,T}(X^\pi_T+\xi) + \int_0^T g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le V(x) = E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]. \qquad \Diamond
\end{aligned}$$

4.3. Characterization

We conclude this section by providing a characterization of an optimal trading strategy and a corresponding optimal model in the framework of the stochastic maximum principle. It dates back to the work of Bismut in the 1970s. The maximum principle has been widely used in the context of expected utility maximization to characterize optimal strategies, see for instance Horst et al. [20]. Applying the perturbation techniques yielding the stochastic maximum principle as developed by Peng [26] to the control problem (2.3) as it is does not give much information on the optimal solution because of the nonlinearity of the operator $\mathcal{E}^g_0$. This is where the dual representation for BSDEs becomes useful, in helping to linearize the problem by transforming it into a robust control problem under a linear operator. In the following we denote by $\partial g^*/\partial a$ and $\partial g^*/\partial b$, when they exist, the derivative of the function $g^*:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ with respect to the first and the second variable, respectively.

Since for every $\pi\in\Pi$ the process X^π is a positive Q-martingale, we can write X^π as

$$X^\pi_t = x + \int_0^t \sigma_u\tilde\pi_u X^\pi_u\,dW^Q_u \tag{4.14}$$

for some predictable process $\tilde\pi$ satisfying $\{\int_0^T |\sigma_u\tilde\pi_u|^2\,du<\infty\} = \{X^\pi_T>0\}$, see [23, Chapter 1]. The next theorem gives a characterization of the optimal model (q^*,β^*) and of the process $\tilde\pi^*$ associated to the optimal strategy π^*.

Theorem 4.8. *Assume that the driver g is strictly convex, satisfies (ADM), (LSC), (NOR), (POS), and (QG). Further assume that $\xi \in L^\infty_+$. Then, for every saddle point $(\pi^*, (\beta^*, q^*))$ there exists a pair (p, k) depending on $\tilde\pi^*, \beta^*$ and q^* such that $p_t\theta_t + p_t q_t^* + k_t = 0$ $P \otimes dt$ -a.s. and which solves the BSDE*

$$dp_t = -(\theta_t p_t + p_t q_t^* + k_t)\tilde\pi_t^* \sigma_t \, dt + k_t \, dW_t^{Q^{q^*}}, \quad p_T = D_{0,T}^{\beta^*} \quad Q^{q^*} \text{ a.s.}$$

Furthermore, g^* is differentiable at (β^*, q^*) and satisfies

$$-\frac{\partial g_t^*}{\partial a}(\beta_t^*, q_t^*) + Y_t = 0 \quad \text{and} \quad -\frac{\partial g_t^*}{\partial b}(\beta_t^*, q_t^*) + Z_t = 0; \quad P \otimes dt\text{-a.s.}, \tag{4.15}$$

where (Y, Z) solves the BSDE

$$dY_t = g(Y_t, Z_t)\, dt - Z_t \, dW_t, \quad Y_T = X_T^{\pi^*} + \xi. \tag{4.16}$$

Proof. By assumptions and Remark 4.6 the control problem admits a saddle point $(\pi^*, (\beta^*, q^*))$, that is,

$$\begin{aligned} V(x) =& E_{Q^{q^*}}\left[D_{0,T}^{\beta^*}(X_T^{\pi^*} + \xi) + \int_0^T D_{0,u}^{\beta^*} g_u^*(\beta_u^*, q_u^*)\, du \right] \\ =& \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D_{0,T}^{\beta}(X_T^{\pi} + \xi) + \int_0^T D_{0,u}^{\beta} g_u^*(\beta_u, q_u)\, du \right]. \end{aligned} \tag{4.17}$$

It follows from (4.17) that $X_T^{\pi^*}$ is Q^{q^*}-integrable. Put

$$Y_t := \operatorname*{ess\,inf}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D_{t,T}^{\beta}(X_T^{\pi^*} + \xi) + \int_t^T D_{t,u}^{\beta} g_u^*(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_t \right], \quad t \in [0, T].$$

By [13, Corollary 4.3], for all $t \in [0, T]$, we have

$$Y_t := E_{Q^{q^*}}\left[D_{t,T}^{\beta^*}(X_T^{\pi^*} + \xi) + \int_t^T D_{t,u}^{\beta^*} g_u^*(\beta_u^*, q_u^*)\, du \,\Big|\, \mathcal{F}_t \right]$$

so that applying martingale representation theorem and Itô's formula, we can find a predictable process Z such that (Y, Z) solves the linear BSDE

$$dY_t = (\beta_t^* Y_t + q_t^* Z_t - g_t^*(\beta_t^*, q_t^*))\, dt - Z_t \, dW_t, \quad Y_T = X_T^{\pi^*} + \xi.$$

Moreover, by [13, Theorem 4.6], for almost every (ω, t), the subgradients $\partial g(\omega, t, Y_t, Z_t)$ with respect to (Y_t, Z_t) contain (β_t^*, q_t^*). Hence, (Y, Z) also solves the BSDE (4.16).

Characterization of $\tilde\pi^*$: For any $\pi \in \Pi$ define

$$Y_t^\pi := E_{Q^{q^*}}\left[D_{t,T}^{\beta^*}(X_T^{\pi} + \xi) + \int_t^T D_{t,u}^{\beta^*} g_u^*(\beta_u^*, q_u^*)\, du \,\Big|\, \mathcal{F}_t \right], \quad t \in [0, T].$$

It follows from the saddle point property that

$$V(x) = \sup_{\pi\in\Pi} Y_0^\pi = Y_0^{\pi^*}.$$

Let $\pi \in \Pi$ be a bounded strategy such that for every $\varepsilon \in (0,1)$, $\pi^* + \varepsilon\pi \in \Pi$ and let $\tilde{\pi}$ be the process associated to π, see (4.14). Then, by optimality of π^*,

$$0 = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(Y_0^{\pi^*+\varepsilon\pi} - Y_0^{\pi^*}\right) = E_{Q^{q^*}}\left[D_{0,T}^{\beta^*}\eta_T\right],$$

where $\eta_t := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(X_t^{\pi^*+\varepsilon\pi} - X_t^{\pi^*}\right)$ solves the SDE

$$\begin{aligned} d\eta_t &= \theta_t\left(\tilde{\pi}_t^*\sigma_t\eta_t + X_t^{\pi^*}\sigma_t\tilde{\pi}_t\right)dt + \left(\tilde{\pi}_t^*\sigma_t\eta_t + X_t^{\pi^*}\sigma_t\tilde{\pi}_t\right)dW_t \\ &= \alpha_t(\theta_t + q_t^*)\,dt + \alpha_t\,dW_t^{Q^{q^*}}, \quad \eta_0 = 0 \quad Q^{q^*}\text{-a.s.} \end{aligned}$$

with $\alpha_t = (\tilde{\pi}_t^*\sigma_t\eta_t + X_t^{\pi^*}\sigma_t\tilde{\pi}_t)$. In fact, this follows by applying the dominated convergence theorem [27, Theorem IV.32], since

$$X_t^{\pi^*+\varepsilon\pi} = x\mathcal{E}\left(\tilde{\pi}^* + \varepsilon\tilde{\pi}dW^Q\right)_t \le x\exp\left(\int_0^t \tilde{\pi}_u^*\,dW_u^Q + \left|\int_0^t \tilde{\pi}_u\,dW_u^Q\right| + \frac{1}{2}\int_0^t(\tilde{\pi}_u^* + \tilde{\pi}_u)^2\,du\right),$$

where $dQ/dP = \mathcal{E}(-\int\theta\,dW)_T$. Let (p,k) be the solution of the linear BSDE with bounded terminal condition

$$dp_t = -(\theta_t p_t + q_t^* p_t + k_t)\tilde{\pi}_t^*\sigma_t\,dt + k_t\,dW_t^{Q^{q^*}}, \quad p_T = D_{0,T}^{\beta^*} \quad Q^{q^*}\text{-a.s.}$$

which is known as the adjoint equation. Observe that since $\beta^* \in \mathcal{D}$, $D_{0,T}^{\beta^*}$ is bounded. Applying Itô's formula to $\eta_t p_t$ yields

$$\eta_t p_t = \int_0^t X_u^{\pi^*}\tilde{\pi}_u\sigma_u(p_u\theta_u + p_u q_u^* + k_u)\,du + \int_0^t\left\{\eta_u k_u + p_u(\tilde{\pi}_u^*\sigma_u\eta_u + X_u^{\pi^*}\sigma_u\tilde{\pi}_u)\right\}dW_u^{Q^{q^*}}. \quad (4.18)$$

Since we cannot ensure that the second term of the left hand side of Equation (4.18) is a true Q^{q^*}-martingale, we introduce the following localization:

$$\tau^n := \inf\left\{t \ge 0 : \left|\int_0^t\left\{\eta_u k_u + p_u(\tilde{\pi}_u^*\sigma_u\eta_u + X_u^{\pi^*}\sigma_u\tilde{\pi}_u)\right\}dW_u^{Q^{q^*}}\right| > n\right\}\wedge T.$$

Hence, taking expectation with respect to Q^{q^*} on both sides of (4.18), we have

$$E_{Q^{q^*}}\left[p_{\tau^n}\eta_{\tau^n}\right] = E_{Q^{q^*}}\left[\int_0^{\tau^n} X_u^{\pi^*}\tilde{\pi}_u\sigma_u(p_u\theta_u + p_u q_u^* + k_u)\,du\right]. \quad (4.19)$$

By definition of $\mathcal{D}$, the family $(D_{0,\tau^n}^{\beta^*})_n$ is dominated by the bounded random variable $e^{\int_0^T(\beta_u^*)^-\,du}$. Moreover, for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\eta_{\tau^n} \le \frac{1}{\varepsilon}(X_{\tau^n}^{\pi^*+\varepsilon\pi} - X_{\tau^n}^{\pi^*}) + \delta \le \frac{1}{\varepsilon}X_{\tau^n}^{\pi^*+\varepsilon\pi} + \delta.$$

Because we can restrict ourselves to subsolutions $(Y,Z) \in \mathcal{A}^u(X_T^\pi + \xi)$ satisfying $Y \ge X^\pi$, we can further estimate η_{τ^n} by

$$\eta_{\tau^n} \le \frac{1}{\varepsilon}Y_{\tau^n}^{\pi^*+\varepsilon\pi} + \delta \le \frac{1}{\varepsilon}E_{Q^{q^*}}\left[D_{0,\tau^n}^{\beta^*}(X_T^{\pi^*+\varepsilon\pi} + \xi) + \int_{\tau^n}^T g_u^*(\beta_u^*, q_u^*)\,du \,\middle|\, \mathcal{F}_{\tau^n}\right] + \delta$$

where the second inequality follows from the same arguments which led to Equation (4.4) in the proof of Theorem 4.1. Hence,

$$\eta_{\tau^n} \leq \frac{1}{\varepsilon} E_{Q^{q^*}} \left[e^{\int_0^T (\beta_u^*)^- \, du} (X_T^{\pi^* + \varepsilon\pi} + \xi) + \int_0^T g_u^*(\beta_u^*, q_u^*) \, du \,|\, \mathcal{F}_{\tau^n} \right] + \delta.$$

Since the right hand side above is Q^{q^*}-uniformly integrable, taking the limit in (4.19) and using dominated convergence theorem and Fatou's lemma give

$$E_{Q^{q^*}} \left[\int_0^T X_u^{\pi^*} \tilde{\pi}_u \sigma_u (p_u \theta_u + p_u q_u^* + k_u) \, du \right] \leq E_{Q^{q^*}} \left[D_{0,T}^{\beta^*} \eta_T \right] = 0,$$

recall that both p and η are Q^{q^*}-a.s. continuous processes. Arguing as above with $-\pi$ instead of π, we have

$$E_{Q^{q^*}} \left[\int_0^T X_u^{\pi^*} \tilde{\pi}_u \sigma_u (p_u \theta_u + p_u q_u^* + k_u) \, du \right] = 0.$$

Thus, since π was taken arbitrary, this leads to

$$p_t \theta_t + p_t q_t^* + k_t = 0 \quad P \otimes dt\text{-a.s},$$

since $Q^{q^*} \sim P$.

Characterization of β^* and q^*: The function g satisfies (LSC) and $(\beta^*, q^*) \in \partial g(Y, Z)$ imply that $(Y, Z) \in \partial g^*(\beta^*, q^*)$, and since g is strictly convex, it holds $\partial g^*(\beta^*, q^*) = \{(Y, Z)\}$ so that by [29, Theorem 25.1], g^* is differentiable at (β^*, q^*). Hence, β^* and q^* are the points verifying

$$-\frac{\partial g^*}{\partial a}(\beta_t^*, q_t^*) + Y_t = 0 \quad \text{and} \quad -\frac{\partial g^*}{\partial b}(\beta_t^*, q_t^*) + Z_t = 0 \quad P \otimes dt\text{-a.s.} \qquad \square$$

5. Link to Conjugate Duality

In this final section we show the inherent link between duality of BSDEs and the theory of conjugate duality in optimization as presented, for instance, in Ekeland and Témam [15]. We will exploit the general method of conjugate duality in convex optimization to study the problem at hands. In Proposition 5.2 below we write the dual problem to (2.3). The main result of this section, Theorem 5.3, shows that even without the condition (QG) which enabled us to have weak compactness, the robust control problem still satisfies a minimax property. Consider the probability measure $Q = Q^\theta$ introduced in Section 2. Recall that $\mathcal{H}^1(Q)$ is the set of Q-martingales X such that $E_Q[\sup_{t \in [0,T]} |X|_t] < \infty$. We introduce the sets

$$\mathcal{C} := \{X_T^\pi : \pi \in \Pi\} \cap \mathcal{H}^1(Q), \quad \mathcal{M} := \{M \in \text{BMO}_{++}(Q) : E_Q[M X_T^\pi] \leq x \text{ for all } \pi \in \Pi\}$$

and $\bar{\mathcal{Q}} := \{q \in \mathcal{L} : \frac{dQ^q}{dP} \in \mathcal{M}\}$. Let us define the perturbation function F on $\mathcal{C} \times \mathcal{C}$ with values in $\mathbb{R}$ by

$$F(X_T^\pi, H) := \mathcal{E}_0^g \left(X_T^\pi + \xi + H \right).$$

For all $H \in \mathcal{C}$ we put

$$u(H) := \sup_{\pi \in \Pi} F(X_T^\pi, H).$$

The space $\mathrm{BMO}(Q)$ can be identified with the dual of the space $\mathcal{H}^1(Q)$. We extent the function F to the Banach space $\mathcal{H}^1(Q)\times\mathcal{H}^1(Q)$ by setting $F(X_T^\pi,H)=-\infty$ whenever H or X_T^π does not belong to $\mathcal{C}$. It holds $u(0)=V(x)$, the value function of the primal control problem. Since $\mathcal{E}_0^g$ is concave increasing, the function u is as well concave increasing, and from $u(0)=V(x)<\infty$ follows that $u(H)<\infty$ for all $H\in\mathcal{C}$. Define the concave conjugate F^* of F on $\mathrm{BMO}(Q)\times\mathrm{BMO}(Q)$ with values in $\bar{\mathbb{R}}$ by

$$F^*(M',M):=\inf_{H,X_T^\pi\in\mathcal{H}^1(Q)}\left\{E_Q\left[M'X_T^\pi\right]+E_Q\left[MH\right]-F(X_T^\pi,H)\right\}.$$

The function F^* is concave and upper semicontinuous. For each $M'\in\mathrm{BMO}(Q)$, put

$$v(M'):=\inf_{M\in\mathrm{BMO}(Q)}\{-F^*(M',M)\}. \tag{5.1}$$

For $M'=0$ Equation (5.1) is the dual problem, and the relation $u(0)\le v(0)$ follows as an immediate consequence of the definition of F^*. Since the functional $\mathcal{E}_0^g$ is increasing and $\mathcal{E}_0^g(0)>-\infty$ we have $u(0)\ge\mathcal{E}_0^g(0)>-\infty$. Hence $v(0)>-\infty$.

Lemma 5.1. *Assume that the driver g defined on $\mathbb{R}_+\times\mathbb{R}^d$ satisfies (*C*ONV), (*L*SC), (*N*OR) and (*P*OS). Then, the function F is $\sigma(\mathcal{H}^1(Q)\times\mathcal{H}^1(Q),\mathit{BMO}(Q)\times\mathit{BMO}(Q))$-upper semicontinuous.*

Proof. See Appendix A. □

For any $M\in\mathcal{M}$, define by $\mathcal{E}_0^*$ the convex conjugate of $\mathcal{E}_0^g$ relative to the dual pair $(\mathcal{H}^1(Q),\mathrm{BMO}(Q))$. It follows from [13, Remark 3.8] that for each $M\in\mathcal{M}$, there exists $(\beta,q)\in\mathcal{D}\times\mathcal{Q}$, with q unique such that $M=D_{0,T}^\beta dQ^q/dP$, and $D_{0,T}^\beta=E[M]$. We put

$$\mathcal{E}_0^*(\beta,q):=\inf_{\{M\in\mathcal{M}:E[M]=D_{0,T}^\beta\}}\mathcal{E}_0^*(M).$$

Proposition 5.2. *Assume that the driver g defined on $\mathbb{R}_+\times\mathbb{R}^d$ satisfies (*C*ONV), (*L*SC), (*N*OR) and (*P*OS). Further assume that $\xi\in L_+^\infty$. Then the dual problem to (2.3) is given by*

$$v(0)=\inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}}\left\{\mathcal{E}_0^*(\beta,q)-E_Q\left[\frac{dQ^q}{dP}D_{0,T}^\beta\xi\right]\right\}-x \tag{5.2}$$

and the primal problem

$$u(0)=\sup_{X_T^\pi\in\mathcal{C}}\inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}}E_{Q^q}\left[D_{0,T}^\beta\frac{dQ}{dP}(X_T^\pi+\xi)+\int_0^T D_{0,u}^\beta g_u^*(\beta_u,q_u)\,du\right].$$

Proof. For every $M\in L^\infty$ one has

$$\begin{aligned}F^*(0,M)&=\inf_{H\in\mathcal{H}^1(Q),X_T^\pi\in\mathcal{C}}\{E_Q\left[MH\right]-F(X_T^\pi,H)\}\\&=\inf_{H'\in\mathcal{H}^1(Q),X_T^\pi\in\mathcal{C}}\left\{E_Q\left[M(H'-X_T^\pi-\xi)-F(X_T^\pi,H'-X_T^\pi-\xi)\right]\right\}.\end{aligned}$$

In fact, $\{H'-X_T^\pi-\xi : X_T^\pi\in\mathcal{C},\ H'\in\mathcal{H}^1(Q)\}\subseteq\mathcal{H}^1(Q)$ and, reciprocally, for any $H\in\mathcal{H}^1(Q)$ we can write $H=H'-x-\xi=H'-X_T^0-\xi$ for some $H'\in\mathcal{H}^1(Q)$. Hence,

$$F^*(0,M)=\inf_{H'\in\mathcal{H}^1(Q)}\{E_Q\left[MH'\right]-\mathcal{E}_0^g(H')\}-\sup_{X_T^\pi\in\mathcal{C}}E_Q\left[M(X_T^\pi+\xi)\right].$$

It is clear that if there exists $X_T^\pi \in \mathcal{C}$ such that $E_Q[MX_T^\pi] > x$, then $F^*(0, M) = -\infty$. Thus, the supremum in Equation (5.1) can by restricted to $\mathcal{M}$, and $F^*(0, M)$ takes the form

$$F^*(0, M) = \mathcal{E}_0^*(M) - E_Q\left[M\xi\right] - x.$$

Therefore, the dual problem (5.1) to the control problem (2.3) is given by

$$\begin{aligned} v(0) &= \inf_{M\in\mathcal{M}} \left\{\mathcal{E}_0^*(M) - E_Q\left[M\xi\right]\right\} - x \\ &= \inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}} \inf_{\{M:E[M]=D^\beta_{0,T}\}} \left\{\mathcal{E}_0^*(M) - E_Q\left[\frac{dQ^q}{dP}D^\beta_{0,T}\xi\right]\right\} - x \\ &= \inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}} \left\{\mathcal{E}_0^*(\beta, q) - E_Q\left[\frac{dQ^q}{dP}D^\beta_{0,T}\xi\right]\right\} - x. \end{aligned} \tag{5.3}$$

Now, let us introduce the following Lagrangian L, which is such that $-L$ is the H-conjugate of the function F, i.e.

$$L(X_T^\pi, M) = \sup_{H\in\mathcal{C}} \left\{F(X_T^\pi, H) - E_Q\left[MH\right]\right\}.$$

It is well known in convex duality theory, see for instance [15], that the following hold:

$$F^*(M', M) = \inf_{X_T^\pi\in\mathcal{C}} \left\{E_Q\left[M'X_T^\pi\right] - L(X_T^\pi, M)\right\}$$

and, since F is $\sigma(\mathcal{H}^1(Q)\times\mathcal{H}^1(Q), \mathrm{BMO}(Q)\times\mathrm{BMO}(Q))$-upper semicontinuous, the Fenchel-Moreau theorem and definition of L yield

$$F(X_T^\pi, H) = \inf_{M\in\mathcal{M}} \left\{E_Q\left[MH\right] + L(X_T^\pi, M)\right\}. \tag{5.4}$$

In particular,

$$v(0) = \inf_{M\in\mathcal{M}} \sup_{X_T^\pi\in\mathcal{C}} \left\{L(X_T^\pi, M)\right\} \quad \text{and} \quad u(0) = \sup_{X_T^\pi\in\mathcal{C}} \inf_{M\in\mathcal{M}} \left\{L(X_T^\pi, M)\right\}.$$

Let $\pi \in \Pi$ and $M \in \mathcal{M}$. By definition of the Laplacian, we have

$$\begin{aligned} L(X_T^\pi, M) &= \sup_{H\in\mathcal{H}^1(Q)} \left\{F(X_T^\pi, H) - E_Q\left[MH\right]\right\} \\ &= \sup_{H'\in\mathcal{H}^1(Q)} \left\{F(X_T^\pi, H' - X_T^\pi - \xi) - E_Q\left[M(H' - X_T^\pi - \xi)\right]\right\} \\ &= \sup_{H'\in\mathcal{H}^1(Q)} \left\{\mathcal{E}_0^g(H') - E_Q\left[MH'\right]\right\} + E_Q\left[M(X_T^\pi + \xi)\right] \\ &= E_Q\left[M(X_T^\pi + \xi)\right] - \mathcal{E}_0^*(M). \end{aligned}$$

But by the proof of [13, Theorem 3.10], the function

$$\alpha_{\min} : M \mapsto \inf_{\{\beta\in\mathcal{D}:E[M]=D^\beta_{0,T}\}} E_{Q^q}\left[\int_0^T D^\beta_{0,u} g_u^*(\beta_u, q_u)\,du\right]$$

is convex and $\sigma(\mathcal{H}^1(Q), \mathrm{BMO(Q)})$-lower semicontinuous; that is, it is the minimal penalty function. Hence, $-\mathcal{E}_0^*(M) = \alpha_{\min}(M)$ and therefore,

$$L(X_T^\pi, M) = \inf_{\{\beta\in\mathcal{D}:E[M]=D^\beta_{0,T}\}} E_{Q^q}\left[D^\beta_{0,T}\frac{dQ}{dP}(X_T^\pi + \xi) + \int_0^T D^\beta_{0,u} g_u^*(\beta_u, q_u)\,du\right].$$

In particular, this implies

$$u(0) = \sup_{X_T^\pi \in \mathcal{C}} \inf_{(\beta,q)\in \mathcal{D}\times\bar{\mathcal{Q}}} E_{Q^q}\left[D_{0,T}^\beta \frac{dQ}{dP}(X_T^\pi + \xi) + \int_0^T D_{0,u}^\beta g_u^*(\beta_u, q_u)\, du \right]. \qquad \square$$

Next, we show that the control problem (2.3) satisfies the minimax property even if we do not assume any growth condition on the generator g. Notice that it does not ensure existence of a saddle point. We refer to [1] for some similar results in robust utility maximization.

Theorem 5.3. *Assume that the driver g satisfies (*Conv*), (*Lsc*), (*Nor*) and (*Pos*). Then, the value functions of the primal problem and dual problem coincide. More precisely, it holds*

$$\inf_{M\in\mathcal{M}} \sup_{X_T^\pi\in\mathcal{C}} \{L(X_T^\pi, (\beta, q))\} = \sup_{X_T^\pi\in\mathcal{C}} \inf_{M\in\mathcal{M}} \{L(X_T^\pi, (\beta, q))\}.$$

Proof. The main argument of the proof is the Fenchel-Rockafellar theorem applied on the Banach space $\mathcal{H}^1(Q)$. By definition $\mathcal{M} = \mathcal{C}^*$, the polar cone of $\mathcal{C}$ with respect to the dual pair $(\mathcal{H}^1(Q), \mathrm{BMO}(Q))$. Moreover, since $\mathcal{C}$ is a cone, $\mathcal{M}$ is the polar of $\mathcal{C}$, i.e. $\mathcal{M} = \mathcal{C}^\circ$. Consider the convex-indicator function

$$\delta_{\mathcal{C}}(H) = \begin{cases} 0 & \text{if} \quad H \in \mathcal{C} \\ \infty & \text{if} \quad H \in \mathcal{H}^1(Q)\setminus \mathcal{C}. \end{cases}$$

We can rewrite u as

$$u(0) = \sup_{H\in\mathcal{H}^1(Q)} \{F(H,0) - \delta_{\mathcal{C}}(H)\}.$$

Since $\mathcal{C}$ is $\sigma(\mathcal{H}^1(Q), \mathrm{BMO}(Q))$-closed (see proof of Lemma 5.1), the function $F(\cdot, 0) - \delta_{\mathcal{C}}(\cdot)$ is concave and $\sigma(\mathcal{H}^1(Q), \mathrm{BMO}(Q))$-upper semicontinuous. Hence, by [28, Corollary 1] we have

$$u(0) = \inf_{M\in\mathrm{BMO}(Q)} \{\delta_{\mathcal{C}}^*(\beta, q) - F^*(0, M)\}.$$

The function $\delta_{\mathcal{C}}$ obeys the conjugacy relation $\delta_{\mathcal{C}}^* = \delta_{\mathcal{C}^\circ} = \delta_{\mathcal{M}}$, see [30, Section 11.E]. Thus,

$$\begin{aligned} u(0) &= \inf_{M\in\mathrm{BMO}(Q)} \{\delta_{\mathcal{M}}(M) - F^*(0, M)\} \\ &= \inf_{M\in\mathcal{M}} \{-F^*(0, M)\} = v(0). \end{aligned}$$

This concludes the proof. $\square$

A. Proofs of Intermediate Results

Proof (of Lemma 3.3). Let (X^1, Y^1, Z^1) and (X^2, Y^2, Z^2) be two elements of $\mathcal{A}(x)$; and $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$. Then, by joint convexity of g, $(\lambda_1 Y^1 + \lambda_2 Y^2, \lambda_1 Z^1 + \lambda_2 Z^2)$ satisfies Equation (2.2) and the terminal condition $\lambda_1 Y_T^1 + \lambda_2 Y_T^2 \le \lambda_1 X_T^1 + \lambda_2 X_T^2 + \xi$ is also satisfied. In addition, since $u^{-1}(E[u(\cdot)])$ is concave, for all $0 \le s \le t \le T$, we have

$$\begin{aligned} u^{-1}(E[u(\lambda_1 Y_t^1 + \lambda_2 Y_t^2) \,|\, \mathcal{F}_s]) &\ge \lambda_1 u^{-1}(E[u(Y_t^1) \,|\, \mathcal{F}_s]) + \lambda_2 u^{-1}(E[u(Y_t^2) \,|\, \mathcal{F}_s]) \\ &\ge \lambda_1 u^{-1}(u(Y_s^1)) + \lambda_2 u^{-1}(u(Y_s^2)) \\ &= \lambda_1 Y_s^1 + \lambda_2 Y_s^2, \end{aligned}$$

where the second inequality comes from the facts that Y^1 and Y^2 are admissible and u^{-1} increasing. Hence because u is increasing, we have

$$E[u(\lambda_1 Y_t^1 + \lambda_2 Y_t^2) \,|\, \mathcal{F}_s] \geq u(\lambda_1 Y_s^1 + \lambda_2 Y_s^2),$$

which implies that $\lambda_1 Y^1 + \lambda_2 Y^2$ is admissible. Put $X^1 = X^{\pi^1}$ and $X^2 = X^{\pi^2}$. The process $\lambda_1 X^1 + \lambda_2 X^2$ is a wealth process, since

$$\lambda_1 X_t^1 + \lambda_2 X_t^2 = x + \int_0^t (\lambda_1 \pi_u^1 + \lambda_2 \pi_u^2)\sigma_u \, dW_u^Q.$$

□

Proof (of Proposition 3.7). First notice that the operator $\mathcal{E}_0^g(\cdot)$ is increasing. Indeed, if $\xi' \leq \xi$ then $\mathcal{A}^r(\xi', g) \subseteq \mathcal{A}^r(\xi, g)$, which implies $\mathcal{E}_0^g(\xi') \leq \mathcal{E}_0^g(\xi)$. Since the sequence $(\xi^n) \subseteq L_+^\infty$ is decreasing, the limit ξ belongs to L_+^∞. By monotonicity, $(\mathcal{E}_0^g(\xi^n))$ is a decreasing sequence, bounded from below by $\mathcal{E}_0^g(\xi)$. Thus, we can define $Y_0 := \lim_{n\to\infty} \mathcal{E}_0^g(\xi^n) \geq \mathcal{E}_0^g(\xi)$. By monotonicity and the condition (NOR), $\mathcal{E}_0^g(\xi) \geq \mathcal{E}_0^g(0) > -\infty$. Theorem 3.4 yields a maximal subsolution $(\bar{Y}^n, \bar{Z}^n) \in \mathcal{A}^r(\xi^n, g)$ with $\bar{Y}_0^n = \mathcal{E}_0^g(\xi^n)$ for all $n \in \mathbb{N}$. We can use the method introduced in the proof of Theorem 3.4 to obtain a pair $(\bar{Y}, \bar{Z}) \in \mathcal{A}^r(\xi, g)$ with

$$Y_0 = \lim_{n\to\infty} \mathcal{E}_0^g(\xi^n) = \bar{Y}_0 = \mathcal{E}_0^g(\xi).$$

The sequence $(\bar{Y}_0^n)$ is not increasing as in the proof of Theorem 3.4 but decreasing. Nevertheless we can obtain an estimate such as that of (3.6) using $\bar{Y}_0^n \leq Y_0^1$. Finally, $\mathcal{E}_0^g(\xi)$ is optimal. In fact, let $(Y, Z) \in \mathcal{A}^r(\xi, g)$ be any subsolution. Since $\xi \leq \xi^n$ for all $n \in \mathbb{N}$, we have $(Y, Z) \in \mathcal{A}^r(\xi^n, g)$. Thus, $Y_0 \leq \mathcal{E}_0^g(\xi^n)$ for all n. Taking the limit as n tends to infinity, we conclude $Y_0 \leq \mathcal{E}_0^g(\xi)$. □

Proof (of Proposition 3.8). Since (g^n) is increasing, $(\mathcal{E}_0^{g^n}(\xi))$ is decreasing and bounded from below by $\mathcal{E}_0^g(\xi)$. Define $Y_0 := \lim_{n\to\infty} \mathcal{E}_0^{g^n}(\xi) \geq \mathcal{E}_0^g(\xi)$. Y_0 is finite since $\mathcal{E}_0^g(\xi) \leq Y_0 \leq \mathcal{E}_0^{g^1}(\xi)$. For all n, there exists $(\bar{Y}^n, \bar{Z}^n) \in \mathcal{A}^r(\xi, g^n)$ such that $\mathcal{E}_0^{g^n}(\xi) = \bar{Y}_0^n$. Then by the method introduced in the proof of Theorem 3.4 we can obtain a candidate $(\bar{Y}, \bar{Z})$, maximal subsolution of the system with parameters g and ξ. The verification that $(\bar{Y}, \bar{Z})$ is indeed an element of $\mathcal{A}^r(\xi, g)$ relies on Fatou's lemma and monotone convergence theorem, since $g^n \uparrow g$. See the proof of [12, Theorem 4.14] for similar arguments. The subsolution $(\bar{Y}, \bar{Z})$ is maximal, since $\bar{Y}_0 = Y_0$. □

Proof (of Lemma 5.1). Let us first show that $\mathcal{C}$ is closed in $\mathcal{H}^1(Q)$. For any sequence $(X_T^n) \subseteq \mathcal{C}$ converging to X_T in $\mathcal{H}^1(Q)$, the process $X_t := E_Q[X_T \,|\, \mathcal{F}_t]$ defines a positive Q-martingale starting at x. By martingale representation theorem, there exists $\nu \in \mathcal{L}^1(Q)$ such that $X_t = x + \int_0^t \nu_u \, dW_u^Q$, but since $\sigma\sigma'$ is of full rank, we can find a predictable process π such that $\pi\sigma = \nu$. Therefore, $dX_t = \pi_t \sigma_t(\theta_t dt + dW_t)$. That is, $X \in \mathcal{C}$.

Now it suffices to show that the function F is $\sigma(\mathcal{H}^1(Q) \times \mathcal{H}^1(Q), \mathrm{BMO}(Q) \times \mathrm{BMO}(Q))$-upper semi-continuous on $\mathcal{C} \times \mathcal{C}$ because the extension to $\mathcal{H}^1(Q) \times \mathcal{H}^1(Q)$ would also be weakly upper semicontinuous. Hence, we need to show that for every $c \geq 0$ the concave level set $\{(\alpha, \gamma) \in \mathcal{C} \times \mathcal{C} : F(\alpha, \gamma) \geq c\}$ is closed in $\mathcal{C} \times \mathcal{C}$. Let $c \geq 0$ be fixed and let us show that $\{\zeta \in \mathcal{H}^1(Q) : \mathcal{E}_0^g(\zeta) \geq c\}$ is $\mathcal{H}^1(Q)$-closed. Let (ζ^n) be a sequence converging in $\mathcal{H}^1(Q)$ to ζ and such that $\mathcal{E}_0^g(\zeta^n) \geq c$ for every $n \in \mathbb{N}$. Put $\eta^n := \sup_{m\geq n} \zeta^m$, $n \in \mathbb{N}$. The sequence (η^n) decreases to ζ and by Proposition 3.7, $(\mathcal{E}_0^g(\eta^n))$ converges to $\mathcal{E}_0^g(\zeta)$ and is decreasing. Hence, since $\mathcal{E}_0^g(\zeta) = \lim_{n\to\infty} \mathcal{E}_0^g(\eta^n) = \inf_n \mathcal{E}_0^g(\eta^n)$, it holds

$$\begin{aligned}\mathcal{E}_0^g(\zeta) &= \inf_{n\in\mathbb{N}} \mathcal{E}_0^g\left(\sup_{m\geq n} \zeta^m\right) \\ &\geq \inf_{n\in\mathbb{N}} \sup_{m\geq n} \mathcal{E}_0^g(\zeta^m) = \limsup_{n\to\infty} \mathcal{E}_0^g(\zeta^n).\end{aligned}$$

Now for every sequence $(\alpha^n, \gamma^n) \subseteq \mathcal{C} \times \mathcal{C}$ converging to $(\alpha, \gamma) \in \mathcal{C} \times \mathcal{C}$ in $\mathcal{H}^1(Q) \times \mathcal{H}^1(Q)$ such that $F(\alpha^n, \gamma^n) \geq c$ for every $n \in \mathbb{N}$ one has

$$\begin{aligned} c &\leq \limsup_{n\to\infty} F(\alpha^n, \gamma^n) = \limsup_{n\to\infty} \mathcal{E}_0^g(\alpha^n + \gamma^n + \xi) \\ &\leq \mathcal{E}_0^g(\alpha + \gamma + \xi) = F(\alpha, \gamma). \end{aligned}$$

This concludes the proof. □

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