Weak Closedness of Monotone Sets of Lotteries and Robust Representation of Risk Preferences

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We prove a closedness result for sets of lotteries that are monotone with respect to first order stochastic dominance and show how it can be applied to obtain robust representations of risk preferences on lotteries with compact support.

Keywords: Risk preferences; robust representations; lotteries with compact support; monotonicity

Introduction

We consider a risk preference given by a total preorder $\succcurlyeq$ on the set $\mathcal{M}_{1,c}$ of probability distributions on $\mathbb{R}$ with compact support, that is, a transitive binary relation such that for all $\mu, \nu \in \mathcal{M}_{1,c}$ one has $\mu \succcurlyeq \nu$ or $\mu \preccurlyeq \nu$ or both. Elements μ of $\mathcal{M}_{1,c}$ are understood as lotteries, and $\mu \succcurlyeq \nu$ means that μ is at least as risky as ν.

The goal of the paper is to provide conditions under which $\succcurlyeq$ has a numerical representation of the form

$$\rho(\mu) = \sup_{l \in L} R(l, \langle l, \mu \rangle), \tag{0.1}$$

where L is the set of all non-increasing continuous functions $l : \mathbb{R} \to \mathbb{R}$, $\langle l, \mu \rangle := \int_{\mathbb{R}} l \, d\mu$ and $R : L \times \mathbb{R} \to [-\infty, +\infty]$ is a function satisfying

(R1) $R(l, s)$ is left-continuous and non-decreasing in s;

(R2) R is quasi-concave in (l, s);

(R3) $R(\lambda l, s) = R(l, s/\lambda)$ for all $l \in L$, $s \in \mathbb{R}$ and $\lambda > 0$;

(R4) $\inf_{s \in \mathbb{R}} R(l^1, s) = \inf_{s \in \mathbb{R}} R(l^2, s)$ for all $l^1, l^2 \in L$;

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(R5) $R^+(l,s) := \inf_{t>s} R(l,t)$ is upper semi-continuous in l with respect to $\sigma(C, \mathcal{M}_{1,c})$, where C denotes the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Relation (0.1) can be viewed as a robust representation of risk. Each $l \in L$ induces a risk order on $\mathcal{M}_{1,c}$ through the affine mapping $\mu \mapsto \langle l, \mu \rangle$. Relation (0.1) takes all this orders into account but gives them different impacts by weighing them according to the risk function R. It follows from (R1) that every mapping $\rho : \mathcal{M}_{1,c} \to [-\infty, \infty]$ of the form (0.1) has the following three properties:

(A1) quasi-convexity;

(A2) monotonicity with respect to first order stochastic dominance;

(A3) lower semicontinuity with respect to the weak topology $\sigma(\mathcal{M}_{1,c}, C)$.

Sufficient conditions for preferences on lotteries to have affine representations go back to von Neumann and Morgenstern [9]. For an overview of subsequent extensions we refer to Fishburn [7]. Representations of the form (0.1) have recently been given by Cerreia-Vioglio [2] and Drapeau and Kupper [6]. The contribution of this paper is that we do not make assumptions on $\succcurlyeq$ involving the topology $\sigma(\mathcal{M}_{1,c}, C)$ since they are technical and difficult to check empirically. Instead, we provide conditions with a certain normative appeal and show that they imply that the sublevel sets of $\succcurlyeq$ are closed in $\sigma(\mathcal{M}_{1,c}, C)$. Similar results are given in Delbaen, Drapeau, and Kupper [5] for preferences satisfying the independence and Archimedean axioms. For automatic continuity and representation results on risk measures defined on spaces of random variables we refer to Cheridito and Li [3, 4] and the references therein.

As an example we discuss Value-at-Risk. It is well-known that as a function of random variables, it is not quasi-convex. But Value-at-Risk only depends on the distribution μ_X of a random variable X, and convex combinations act differently on distributions than on random variables. Except for trivial cases, one has $\lambda\mu_X + (1-\lambda)\mu_Y \neq \mu_{\lambda X + (1-\lambda)Y}$. It turns out that as a function on $\mathcal{M}_{1,c}$, Value-at-risk is quasi-convex, $\sigma(\mathcal{M}_{1,c}, C)$-lower semicontinuous and monotone with respect to first order stochastic dominance. As a consequence, it can be represented in the form (0.1); see Example 1.4 below.

The rest of the paper is organized as follows. In Section 1 we introduce the conditions we need to show that $\succcurlyeq$ has a representation of the form (0.1) and state the paper's main results, Theorems 1.1 and 1.2. Section 2 contains a discussion of the weak topology $\sigma(\mathcal{M}_{1,c}, C)$ and the proof of Theorem 1.1.

1 Robust representation of risk preferences on lotteries

To formulate the conditions (C1)–(C3) below we need the following notation:

- By $\mathcal{M}_1$ we denote the set of all probability distributions on $\mathbb{R}$. For $\mu \in \mathcal{M}_1$, we set $F_\mu(x) := \mu(-\infty, x]$ as well as

$$\mu_* := \sup\{x \in \mathbb{R} : F_\mu(x) = 0\} \quad \text{and} \quad \mu^* := \inf\{x \in \mathbb{R} : F_\mu(x) = 1\},$$

  where $\sup \emptyset := -\infty$ and $\inf \emptyset := +\infty$.

- By $\trianglerighteq$ we denote first order stochastic dominance on $\mathcal{M}_1$, that is,

$$\mu \trianglerighteq \nu \quad :\Leftrightarrow \quad F_\mu(x) \leq F_\nu(x) \text{ for all } x \in \mathbb{R}.$$

- For $m \in \mathbb{R}$ and $\mu \in \mathcal{M}_1$, we denote by $T_m\mu$ the shifted distribution given by $T_m\mu(A) = \mu(A-m)$.

To show that the risk preference $\succcurlyeq$ has a representation of the form (0.1) we assume that for each $\nu \in \mathcal{M}_{1,c}$ the sublevel set

$$\mathcal{S}_\nu := \{\mu \in \mathcal{M}_{1,c} : \mu \preccurlyeq \nu\}$$

satisfies the following conditions:

(C1) $\mathcal{S}_\nu$ is convex;

(C2) If $T_m\mu \in \mathcal{S}_\nu$ for all $m > 0$, then $\mu \in \mathcal{S}_\nu$;

(C3) If $\mu \in \mathcal{M}_{1,c}$ has the property that for every $\lambda \in [0,1)$ and each $\eta \in \mathcal{M}_1$ with $\eta_* \geq \mu^*$ and $\eta^* = +\infty$ one has $\lambda\mu + (1-\lambda)\,\eta \trianglerighteq \tau$ for some $\tau \in \mathcal{S}_\nu$, then $\mu \in S_\nu$.

First, let us note that (C3) implies

(C4) $\mu \preccurlyeq \nu$ whenever $\mu \trianglerighteq \nu$,

which is a standard assumption. It just means that "more is better" or "more is less risky". Assumption (C1) is also standard and corresponds to the idea that "averages are better than extremes" or "diversification should not increase the risk". As for (C2) and (C3), they allow us to deduce that all sublevel sets $\mathcal{S}_\nu$ are closed in $\sigma(\mathcal{M}_{1,c}, C)$, which is needed to derive a representation of the form (0.1). But $\sigma(\mathcal{M}_{1,c}, C)$-closedness is a technical condition which is difficult to check. On the other hand, (C2) and (C3) have a certain normative appeal and are much easier to test. Indeed, (C2) is a one-dimensional assumption and means that if a lottery ν is at least as risky as μ shifted to the right by every arbitrarily small amount, then ν is also at least as risky as μ. To put (C3) into perspective, we note that it is considerably weaker than the condition

(C3') If for $\mu \in \mathcal{M}_{1,c}$ there exists an $\eta \in \mathcal{M}_1$ such that for all $\lambda \in [0,1)$ one has $\lambda\mu + (1-\lambda)\,\eta \trianglerighteq \tau$ for some $\tau \in \mathcal{S}_\nu$, then $\mu \in S_\nu$;

which is a stronger version of the directional closedness assumption

(C3") If $\mu, \eta \in \mathcal{M}_{1,c}$ are such that $\lambda\mu + (1-\lambda)\,\eta \in S_\nu$ for all $\lambda \in [0,1)$, then $\mu \in S_\nu$.

Remark 1.3 below shows that a subset $\mathcal{A}$ of a Banach lattice $(E, \geq)$ is norm-closed if it satisfies (C3") and the monotonicity condition

(C4') $\mu \geq \tau \in \mathcal{A}$ implies $\mu \in \mathcal{A}$.

However, $(\mathcal{M}_{1,c}, \trianglerighteq)$ is only a convex set with a partial order and the topology $\sigma(\mathcal{M}_{1,c}, C)$ is not metrizable; see Remark 2.1. For our proof of Theorem 1.1 to work, conditions like (C3") and (C4') are not enough. It needs (C2) and (C3).

Theorem 1.1. *Every subset $\mathcal{A}$ of $\mathcal{M}_{1,c}$ satisfying* (C2) *and* (C3) *is $\sigma(\mathcal{M}_{1,c}, C)$-closed.*

The proof is given in Section 2. As a consequence one obtains the following

Theorem 1.2. *If all sublevel sets of $\succcurlyeq$ satisfy* (C1)–(C3)*, then $\succcurlyeq$ has a numerical representation $\rho : \mathcal{M}_{1,c} \to [-\infty, \infty]$ satisfying* (A1)–(A3)*. Moreover, for every such ρ there exists a unique risk function R with the properties* (R1)–(R5) *such that* (0.1) *holds.*

Proof. By Theorem 1.1, the sublevel sets of $\succcurlyeq$ are closed in $\sigma(\mathcal{M}_{1,c}, C)$. Since they are also convex and monotone with respect to $\trianglerighteq$, the theorem follows from Drapeau and Kupper [6, Theorem 3.5]. $\square$

Remark 1.3. A subset $\mathcal{A}$ of a Banach lattice $(E, \geq)$ satisfying (C3") and (C4') is norm-closed. Indeed, if x_n is a sequence in $\mathcal{A}$ converging to $x \in E$, one can pass to a subsequence and assume $\|x_n - x\| \leq 2^{-n}/n$. For $y := x + \sum_{k=1}^{\infty} k(x_k - x)^+$ and $\lambda \in [0, 1)$, one then has

$$\lambda x + (1-\lambda)y = x + (1-\lambda)\sum_{k=1}^{\infty} k(x_k - x)^+ \geq x + (1-\lambda)n(x_n - x)^+ \geq x_n$$

for all $n \geq 1/(1-\lambda)$. Hence, $\lambda x + (1-\lambda)y \in \mathcal{A}$ for each $\lambda \in [0, 1)$, from which one obtains $x \in \mathcal{A}$. ♦

Example 1.4. Value-at-Risk is a risk measure widely used in the banking industry. For a random variable X and a level $\alpha \in (0, 1)$, it is defined by

$$V@R_\alpha(X) = \inf\{x \in \mathbb{R} : P[X + x < 0] \leq \alpha\},$$

and gives the minimal amount of money which has to be added to X to keep the probability of default below α. It is well-known that the sublevel sets of $V@R_\alpha$ are not convex; see for instance, Artzner, Delbaen, Eber, and Heath [1] or Föllmer and Schied [8]. However, it depends on X only through its distribution. So it can be defined on $\mathcal{M}_{1,c}$ by

$$V@R_\alpha(\mu) = -q_\mu^+(\alpha), \tag{1.1}$$

where q_μ^+ is the right-quantile function of μ given by

$$q_\mu^+ := \sup\{x \in \mathbb{R} : F_\mu(x) \leq \alpha\}.$$

As subsets of $\mathcal{M}_{1,c}$, the sublevel sets are convex. Moreover, it can easily be checked that they satisfy (C2) and (C3). So it follows from Theorem 1.2 that (1.1) has a robust representation of the form (0.1). Indeed, one has

$$V@R_\alpha(\mu) = \sup_{l \in L} -l^{-1}\left(\frac{\langle l, \mu\rangle - \alpha l(-\infty)}{1-\alpha}\right) = -\inf_{l \in L} l^{-1}\left(\frac{\langle l, \mu\rangle - \alpha l(-\infty)}{1-\alpha}\right)$$

where l^{-1} is the left-inverse of l; see Drapeau and Kupper [6]. ◊

The two following examples show that none of the conditions (C2) and (C3) can be dropped from the assumptions of Theorem 1.1.

Example 1.5. The set

$$\mathcal{A} := \{\mu \in \mathcal{M}_{1,c} : \mu_* > 0\}.$$

is clearly not $\sigma(\mathcal{M}_{1,c}, C)$-closed since $\delta_{1/n} \in \mathcal{A}$ converges in $\sigma(\mathcal{M}_{1,c}, C)$ to $\delta_0 \notin \mathcal{A}$. However, it fulfills condition (C3). Indeed, if μ is an element of $\mathcal{M}_{1,c}$ such that for all $\lambda \in [0, 1)$ and $\eta \in \mathcal{M}_1$ with $\eta_* \geq \mu^*$ and $\eta^* = +\infty$, one has $\lambda\mu + (1-\lambda)\eta \trianglerighteq \tau$ for some $\tau \in \mathcal{A}$, then $\mu_* > 0$, and therefore, $\mu \in \mathcal{A}$. By Theorem 1.1, $\mathcal{A}$ cannot fulfill condition (C2), which can also be seen directly by observing that $T_m\delta_0 \in \mathcal{A}$ for all $m > 0$ and $\delta_0 \notin \mathcal{A}$. ◊

Example 1.6. Consider the set

$$\mathcal{A} := \left\{ \mu \in \mathcal{M}_{1,c} : \mu^* \geq 2 \text{ and } \mu \trianglerighteq \left(1 - \frac{1}{n}\right)\delta_0 + \frac{1}{n}\delta_1 \text{ for some } n \geq 1 \right\}.$$

It is not $\sigma(\mathcal{M}_{1,c}, C)$-closed since $(1-1/n)\delta_0 + 1/n\delta_2 \in \mathcal{A}$ converges in $\sigma(\mathcal{M}_{1,c}, C)$ to $\delta_0 \notin \mathcal{A}$. It can easily be seen that it fulfills (C2). Indeed, if $T_m\mu \in \mathcal{A}$ for all $m > 0$, then $\mu^* \geq 2$ and $\mu_* \geq 0$. Hence, $\mu \trianglerighteq (1-1/n)\delta_0 + 1/n\delta_1$ for some $n \geq 1$, and thus, $\mu \in \mathcal{A}$. It follows from Theorem 1.1 that (C3) cannot hold. In fact, δ_0 has the property that for all $\lambda \in [0,1)$ and $\eta \in \mathcal{M}_1$ satisfying $\eta_* \geq \delta_0^* = 0$ and $\eta^* = +\infty$, one can find a $\tau \in \mathcal{A}$ such that $\lambda\delta_0 + (1-\lambda)\eta \trianglerighteq \tau$. However, $\delta_0 \notin \mathcal{A}$ since $\delta_0^* = 0 < 2$. $\diamondsuit$

2 Weak closedness of monotone sets of lotteries

Before giving the proof of Theorem 1.1, we compare the topology $\sigma(\mathcal{M}_{1,c}, C)$ to $\sigma(\mathcal{M}_{1,c}, C_b)$, where C_b denotes the space of all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Remark 2.1. It is well-known that the topology $\sigma(\mathcal{M}_1, C_b)$, and therefore also $\sigma(\mathcal{M}_{1,c}, C_b)$, is generated by the Lévy metric

$$d_L(\mu, \nu) := \inf \{\varepsilon > 0 : F_\mu(x - \varepsilon) - \varepsilon \leq F_\nu(x) \leq F_\mu(x + \varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\}.$$

But $\sigma(\mathcal{M}_{1,c}, C)$ is finer than $\sigma(\mathcal{M}_{1,c}, C_b)$, which can easily be seen from the fact that $(1-1/n)\delta_0 + \delta_n/n$ converges to δ_0 in $\sigma(\mathcal{M}_{1,c}, C_b)$ but not in $\sigma(\mathcal{M}_{1,c}, C)$. Moreover, in contrast to $\sigma(\mathcal{M}_{1,c}, C_b)$, $\sigma(\mathcal{M}_{1,c}, C)$ is not metrizable. Indeed, if one assumes that $\sigma(\mathcal{M}_{1,c}, C)$ is generated by a metric. Then for every ball $B_{1/n}(\nu)$ around a fixed $\nu \in \mathcal{M}_{1,c}$, there exist functions $u_1^n, \ldots, u_{i_n}^n$ in $C \setminus \{0\}$ such that

$$U_n := \{\mu : |\langle u_i^n, \mu - \nu\rangle| \leq 1,\, i = 1, \ldots, i_n\} \subset B_{1/n}(\nu).$$

By shifting, one can assume that $\nu^* = 0$. Define the function $u \in C$ by $u(x) = 0$ for $x \leq 0$. For $m = 1, 2, \ldots$, set

$$u(m) = \max_{1 \leq n \leq m} \max_{1 \leq i \leq i_n} (|2u_i^n(m)| \vee m),$$

and interpolate linearly so that it becomes a continuous function $u : \mathbb{R} \to \mathbb{R}$. There must be an n such that

$$U_n \subset B_{1/n}(\nu) \subset \{\mu : |\langle u, \mu - \nu\rangle| \leq 1/2\}. \tag{2.1}$$

Choose $m \geq n$ such that

$$\frac{1}{u(m)} |\langle u_i^n, \nu\rangle| \leq 1/2 \quad \text{for all } i = 1, \ldots, i_n.$$

Set $\lambda = 1/u(m)$ and $\mu = \lambda\delta_m + (1-\lambda)\nu$. Then

$$|\langle u_i^n, \mu - \nu\rangle| = \lambda\, |\langle u_i^n, \delta_m - \nu\rangle| \leq \frac{|u_i^n(m)|}{u(m)} + \lambda\, |\langle u_i^n, \nu\rangle| \leq 1$$

for all $i = 1, \ldots, i_n$. So μ is in U_n but at the same time,

$$\langle u, \mu - \nu\rangle = \lambda\, \langle u, \delta_m - \nu\rangle = 1,$$

a contradiction to (2.1). $\blacklozenge$

Proof of Theorem 1.1

Assume (μ_α) is a net in $\mathcal{A}$ converging to some $\mu \in \mathcal{M}_{1,c}$ in $\sigma(\mathcal{M}_{1,c}, C)$. Fix $m > 0$, $\lambda \in [0, 1)$ and $\eta \in \mathcal{M}_1$ such that $T_m\mu^* \leq \eta_*$ and $\eta^* = +\infty$. Note that

$$\lambda F_\mu(x - m) = 0 \leq F_\alpha(x) \quad \text{for all } x < \mu_* + m \text{ and every } \alpha. \tag{2.2}$$

Set $b := (1 - \lambda) \wedge (m/2)$ and $c := F_\mu(\mu_* + b) > 0$. Since $\mu_\alpha \to \mu$ in $\sigma(\mathcal{M}_{1,c}, C_b)$, there exists α_0 such that

$$F_\mu(x - bc) - bc \leq F_\alpha(x) \quad \text{for all } x \in \mathbb{R} \text{ and } \alpha \geq \alpha_0.$$

For $x \geq \mu_* + m$, one has $F_\mu(x - bc) \geq c$, and therefore,

$$\lambda F_\mu(x - m) \leq \lambda F_\mu(x - bc) \leq F_\mu(x - bc) - bc \leq F_\alpha(x) \quad \text{for all } \alpha \geq \alpha_0. \tag{2.3}$$

It follows from (2.2)–(2.3) that

$$\lambda F_\mu(x - m) + (1 - \lambda) F_\eta(x) \leq F_\alpha(x) \quad \text{for all } \alpha \geq \alpha_0 \text{ and } x < \mu^* + m.$$

Now choose a non-negative function $u \in C$ such that

$$u(x) = 0 \quad \text{for } x \leq \mu^* \quad \text{and} \quad u(x) \geq \frac{1}{(1 - \lambda)\Big(1 - F_\eta(x)\Big)} \quad \text{for } x \geq \mu^* + m.$$

There exists an $\alpha \geq \alpha_0$ such that

$$|\langle u, \mu_\alpha - \mu \rangle| < 1,$$

which implies

$$\lambda F_\mu(x - m) + (1 - \lambda) F_\eta(x) \leq F_\alpha(x) \quad \text{for all } x \geq m + \mu^*.$$

Indeed, if there existed an $x_0 \geq \mu^* + m$ such that

$$\lambda F_\mu(x_0 - m) + (1 - \lambda) F_\eta(x_0) > F_\alpha(x_0),$$

it would follow that

$$\langle u, \mu_\alpha - \mu \rangle = \int u \, d\mu_\alpha \geq u(x_0)(1 - \lambda)(1 - F_\eta(x_0)) \geq 1,$$

a contradiction. So we have shown that

$$\lambda T_m \mu + (1 - \lambda)\eta \trianglerighteq \mu_\alpha.$$

It follows from (C3) that $T_m\mu \in \mathcal{A}$ for all $m > 0$, which by (C2), implies $\mu \in \mathcal{A}$. □

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