

# Representation of increasing convex functionals with countably additive measures

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## Abstract

We derive two types of representation results for increasing convex functionals in terms of countably additive measures. The first is a max-representation of functionals defined on spaces of real-valued continuous functions and the second a sup-representation of functionals defined on spaces of real-valued measurable functions.

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## 1 Introduction

Let  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be an increasing convex functional on a linear space of functions  $f : \Omega \rightarrow \mathbb{R}$ . More precisely,  $\phi$  is convex and satisfies  $\phi(f) \geq \phi(g)$  for  $f \geq g$ , where the second inequality is understood pointwise. By  $I(\phi)$  we denote the algebraic interior of the effective domain  $\text{dom } \phi := \{f \in X : \phi(f) < +\infty\}$ ; that is,  $I(\phi)$  consists of all  $f \in \text{dom } \phi$  with the property that for every  $g \in X$ , there is an  $\varepsilon > 0$  such that  $f + \lambda g \in \text{dom } \phi$  for all  $0 \leq \lambda \leq \varepsilon$ .

If  $X$  is a linear space of bounded measurable functions on a measurable space  $(\Omega, \mathcal{F})$  containing all indicator functions  $1_A$ ,  $A \in \mathcal{F}$ , it follows from standard convex duality arguments (see Section 2) that

$$\phi(f) = \max_{\mu \in ba^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_X^*(\mu)) \quad \text{for all } f \in I(\phi), \quad (1.1)$$

where  $ba^+(\mathcal{F})$  is the set of all finitely additive measure  $\mu$  on  $\mathcal{F}$  satisfying  $\mu(\Omega) < \infty$ ,  $\langle f, \mu \rangle$  denotes the integral  $\int f d\mu$ , and  $\phi_X^*$  is the convex conjugate of  $\phi$ , given by

$$\phi_X^*(\mu) := \sup_{f \in X} (\langle f, \mu \rangle - \phi(f)). \quad (1.2)$$

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In applications, a representation like (1.1) is often more useful if it is in terms of countably instead of finitely additive measures. This paper provides such representations under different assumptions.

For a non-empty set  $\Omega$ , we call a linear subspace  $X$  of  $\mathbb{R}^\Omega$  a Stone vector lattice if for all  $f, g \in X$ , the point-wise minima  $f \wedge g$  and  $f \wedge 1$  also belong to  $X$ . By  $\sigma(X)$  we denote the smallest  $\sigma$ -algebra on  $\Omega$  making all functions  $f \in X$  measurable with respect to the Borel  $\sigma$ -algebra on  $\mathbb{R}$  and by  $ca^+(X)$  all (countably additive) measures on  $\sigma(X)$  satisfying  $\langle f, \mu \rangle < \infty$  for every  $f \in X^+ := \{g \in X : g \geq 0\}$ .  $\phi_X^* : ca^+(X) \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined as in (1.2). If a sequence  $(f^n)$  in  $X$  converges pointwise from above to  $f \in X$ , we write  $f^n \downarrow f$ . Analogously,  $f^n \uparrow f$  means pointwise convergence from below.

The following proposition is a non-linear extension of the Daniell–Stone theorem (see e.g. Theorem 4.5.2 in [7]) and provides context to our main results, Theorems 1.3 and 1.7 below. All proofs are given in Sections 2 and 3.

**Proposition 1.1** *Let  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be an increasing convex functional on a Stone vector lattice  $X$  over a non-empty set  $\Omega$ . Then the implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v)  $\Rightarrow$  (vi) hold among the conditions*

- (i) *There exists an  $f \in I(\phi)$  such that  $\phi(f^n) \downarrow \phi(f)$  for every sequence  $(f^n)$  in  $X$  satisfying  $f^n \downarrow f$*
- (ii)  *$\phi(f^n) \downarrow \phi(f)$  for each  $f \in I(\phi)$  and every sequence  $(f^n)$  in  $X$  satisfying  $f^n \downarrow f$*
- (iii) *For each  $f \in I(\phi)$  and every sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$  there exists an  $\varepsilon > 0$  such that  $\phi(f + \varepsilon f^n) \downarrow \phi(f)$*
- (iv)  *$\phi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu))$  for all  $f \in I(\phi)$*
- (v)  *$\phi(f) = \sup_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu))$  for all  $f \in I(\phi)$*
- (vi)  *$\phi(f^n) \uparrow \phi(f)$  for each  $f \in I(\phi)$  and every sequence  $(f^n)$  in  $X$  satisfying  $f^n \uparrow f$ .*

We are interested in representations of the form (iv) and (v). If  $\phi$  is real-valued and linear, (i) is Daniell’s condition [6] and equivalent to each of (ii), (iii) and (vi). However, for an increasing convex  $\phi$ , (i)–(iii) do not necessarily follow from (vi). Also, in general (iii) is weaker than (ii), and there exist examples which do not satisfy any of the conditions (i)–(vi). These points are illustrated in the following

**Example 1.2** Consider the Stone vector lattice  $l^\infty$  of all bounded functions  $f : \mathbb{N} \rightarrow \mathbb{R}$ , where we use the convention  $\mathbb{N} = \{1, 2, \dots\}$ . Denote by  $ca_1^+(\mathbb{N})$  the set of all probability measures on  $\mathbb{N}$  and by  $ba_1^+(\mathbb{N})$  the set of all finitely additive probability measures on  $\mathbb{N}$ , that is, all finitely additive measures  $\mu$  on  $\mathbb{N}$  satisfying  $\mu(\mathbb{N}) = 1$ .

**1.**  $s(f) := \sup_m f(m)$  defines an increasing convex functional  $s : l^\infty \rightarrow \mathbb{R}$  which clearly fulfills (vi). It can easily be checked that the convex conjugate of  $s$  is  $s_{l^\infty}^*(\mu) = 0$  if  $\mu$  belongs to  $ba_1^+(\mathbb{N})$  and  $s_{l^\infty}^*(\mu) = \infty$  for all  $\mu \in ca^+(\mathbb{N}) \setminus ba_1^+(\mathbb{N})$ . One obviously has

$$s(f) = \sup_{\mu \in ca_1^+(\mathbb{N})} \langle f, \mu \rangle, \tag{1.3}$$

and it follows from (1.1) that

$$s(f) = \max_{\mu \in ba_1^+(\mathbb{N})} \langle f, \mu \rangle. \quad (1.4)$$

(1.3) is of the form (v). Moreover, for all  $f \in l^\infty$  attaining their supremum, the supremum in (1.3) is attained by a Dirac measure. But if  $f \in l^\infty$  does not attain its supremum, then  $s(f)$  cannot be written in the form (iv). So  $s$  satisfies (v)–(vi) but not (i)–(iv).

**2.**

$$p(f) = \sup_{\mu \in ca^+(\mathbb{N})} \langle f, \mu \rangle \quad (1.5)$$

defines an increasing convex functional  $p : l^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$  mapping  $f$  to 0 or  $+\infty$  depending on whether  $s(f) \leq 0$  or  $s(f) > 0$ . So  $f$  belongs to  $I(p)$  if and only if  $s(f) < 0$ , in which case the supremum in (1.5) is attained. It is easy to see that  $p$  fulfills (iii) but not (ii). So by Proposition 1.1, it satisfies (iii)–(vi) but violates (i)–(ii).

**3.** Now pick an increasing  $f \in l^\infty$  that does not attain its supremum, and choose a  $\mu \in ba_1^+(\mathbb{N})$  which maximizes (1.4). Then one has for all  $n$ ,

$$s(f) = \langle f1_{[1,n]}, \mu \rangle + \langle f1_{(n,\infty)}, \mu \rangle \leq f(n)\mu[1, n] + s(f)(1 - \mu[1, n]).$$

It follows that  $\mu[1, n] = 0$  for all  $n$ . So the positive linear functional  $l : l^\infty \rightarrow \mathbb{R}$ , given by  $l(f) := \langle f, \mu \rangle$ , satisfies  $l(1_{[1,n]}) = 0 < l(1) = 1$ , showing that it violates condition (vi), and therefore also (i)–(v).

In the following we introduce four conditions, called (A), (B), (C) and (D), for an increasing convex functional  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  on a Stone vector lattice  $X$  of functions  $f : \Omega \rightarrow \mathbb{R}$  on a topological space  $\Omega$ . If  $X$  consists of continuous functions, (A) and (B) both imply a max-representation like (iv). From each of (C) and (D) we derive a sup-representation similar to (v) in the case where  $X$  is the set of all bounded measurable functions on a Hausdorff space  $\Omega$  equipped with its Borel  $\sigma$ -algebra.

**(A)** For all  $f \in I(\phi)$  and every sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$ , there exists an  $\varepsilon > 0$  such that for each  $\delta > 0$ , there are  $m \in \mathbb{N}$ ,  $g \in \mathbb{R}_+^\Omega$  and an increasing convex function  $\hat{\phi} : Y \rightarrow \mathbb{R}$  on a convex subset  $Y \subseteq \mathbb{R}_+^\Omega$  containing  $\{0, f^m g, (\varepsilon - g)^+, \varepsilon f^n : n \geq m\}$  so that

- (i)  $\{g < \varepsilon\}$  is relatively compact
- (ii)  $\hat{\phi}(f^m g) \leq \delta$  and
- (iii)  $\hat{\phi}(\varepsilon f^n) \geq \phi(f + \varepsilon f^n) - \phi(f)$  for all  $n \geq m$ .

**(B)** For all  $f \in I(\phi)$  and every sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$ , there exist functions  $g, g^1, g^2, \dots$  in  $\mathbb{R}_+^\Omega$  and numbers  $m, m^1, m^2, \dots$  in  $\mathbb{N}$  together with an increasing convex function  $\hat{\phi} : Y \rightarrow \mathbb{R}$  on a convex subset  $Y \subseteq \mathbb{R}_+^\Omega$  containing  $\{0, f^n/m, g, g^n : n \geq m\}$  so that

- (i)  $\{f^m > g/n\}$  is relatively compact and contained in  $\{m^n g^n \geq 1\}$  for all  $n \geq m$
- (ii)  $\hat{\phi}(0) = 0$
- (iii)  $\hat{\phi}(f^n/m) \geq \phi(f + f^n/m) - \phi(f)$  for all  $n \geq m$ .

**Theorem 1.3** *Let  $X$  be a Stone vector lattice  $X$  of continuous functions  $f : \Omega \rightarrow \mathbb{R}$  on a topological space  $\Omega$  and  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  an increasing convex functional satisfying at least one the conditions (A) or (B). Then*

$$\phi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu)) \quad \text{for all } f \in I(\phi).$$

As a special case of Theorem 1.3, one obtains the following variant of the Daniell–Stone theorem:

**Corollary 1.4** *If  $X$  is a Stone vector lattice of continuous functions  $f : \Omega \rightarrow \mathbb{R}$  on a topological space  $\Omega$ , then every positive linear functional  $\phi : X \rightarrow \mathbb{R}$  satisfying (A) or (B) is of the form  $\phi(f) = \langle f, \mu \rangle$ ,  $f \in X$ , for a measure  $\mu \in ca^+(X)$ .*

In various situations, a measure on a  $\sigma$ -algebra  $\mathcal{F}$  of subsets of a topological space  $\Omega$  can be shown to possess regularity properties. Let us call a finite measure  $\mu$  on  $\mathcal{F}$  closed regular if

$$\mu(A) = \sup \{ \mu(B) : B \in \mathcal{F}, B \text{ is closed and } B \subseteq A \} \quad \text{for all } A \in \mathcal{F}$$

and regular if

$$\mu(A) = \sup \{ \mu(B) : B \in \mathcal{F}, B \text{ is closed, compact and } B \subseteq A \} \quad \text{for all } A \in \mathcal{F}.$$

If  $X$  is a Stone vector lattice of real-valued functions containing the constant functions, then every measure  $\mu \in ca^+(X)$  is finite. Moreover, standard arguments (see Section 2 for details) yield the following:

**Proposition 1.5** *Let  $X$  be a family of continuous functions  $f : \Omega \rightarrow \mathbb{R}$  on a topological space  $\Omega$ . Then every finite measure  $\mu$  on  $\sigma(X)$  is closed regular. Furthermore, if  $\mu$  is a finite measure on  $\sigma(X)$  and there exists a sequence  $(K_n)$  of compact sets in  $\sigma(X)$  such that  $\mu(K_n) \rightarrow \mu(\Omega)$ , then  $\mu$  is regular.*

## Examples 1.6

### 1. (Tightness conditions)

Let  $\phi : C_b \rightarrow \mathbb{R} \cup \{+\infty\}$  be an increasing convex functional on the set  $C_b$  of all bounded continuous functions  $f : \Omega \rightarrow \mathbb{R}$  on a topological space  $\Omega$ . Assume  $V$  is a linear space containing all functions of the form  $f1_K$  and  $f1_{K^c}$  for  $f \in C_b$  and  $K$  a compact subset of  $\Omega$ . If  $\phi$  can be extended to an increasing convex  $\psi : V \rightarrow \mathbb{R} \cup \{+\infty\}$  with the property that for every  $f \in I(\phi)$ , there exists a  $\delta > 0$  and a sequence  $(K_n)$  of compact sets such that

$$\psi(f + \delta 1_{K_n^c}) \rightarrow \psi(f), \tag{1.6}$$

then for every  $f \in I(\phi)$  and each sequence  $(f^n) \in C_b^+$  satisfying  $f^n \downarrow 0$ , there exists an  $\varepsilon > 0$  such that  $\phi(f + \varepsilon) < +\infty$  and

$$\psi(f + \varepsilon f^1 1_{K_n^c}) - \psi(f) \leq \psi(f + \varepsilon \|f^1\|_\infty 1_{K_n^c}) - \psi(f) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows that condition (A) holds with  $\hat{\phi}(h) = \phi(f + h) - \phi(f)$ , and one obtains from Theorem 1.3 that

$$\phi(f) = \max_{\mu \in ca^+(C_b)} \langle f, \mu \rangle - \phi_{C_b}^*(\mu) \quad \text{for all } f \in I(\phi). \tag{1.7}$$

If  $\Omega$  is Hausdorff, all compact sets  $K \subseteq \Omega$  are closed and therefore, belong to the Borel  $\sigma$ -algebra  $\mathcal{F}$ . So in this case, if  $\phi : B_b \rightarrow \mathbb{R}$  is an increasing convex functional defined on the space  $B_b$  of all bounded measurable functions  $f : \Omega \rightarrow \mathbb{R}$  with the property that for every constant  $M \geq 1$ , there exists a sequence  $(K_n)$  of compact sets such that

$$\phi(M1_{K_n^c}) \rightarrow \phi(0), \quad (1.8)$$

one deduces as in the proof of (i)  $\Rightarrow$  (ii) of Proposition 1.1 that  $\phi$  satisfies condition (1.6), and as a consequence, is representable as (1.7). If in addition,  $\phi$  has the translation property:  $\phi(f + m) = \phi(f) + m$  for all  $f \in B_b$  and  $m \in \mathbb{R}$ , then (1.8) holds if and only if for every  $M \geq 1$ , there exists a sequence of compacts  $(K_n)$  such that

$$\phi(-M1_{K_n}) \rightarrow \phi(-M).$$

This is slightly weaker than the tightness condition used in Proposition 4.28 of [9] to derive a max-representation for convex risk measures. Note that if  $\phi$  has the translation property, then  $\phi_{C_b}^*(\mu) = +\infty$  for  $\mu \in ca^+(C_b) \setminus ca_1^+(C_b)$ . Consequently, (1.7) reduces to a maximum over probability measures:

$$\phi(f) = \max_{\mu \in ca_1^+(C_b)} (\langle f, \mu \rangle - \phi^*(\mu)), \quad f \in C_b. \quad (1.9)$$

In the special case where  $\Omega$  is a metric space,  $C_b$  generates the Borel  $\sigma$ -algebra  $\mathcal{F}$ , and for every compact set  $K_n$ , there exists a sequence  $(h^{m,n})$  of  $[0, 1]$ -valued functions in  $C_b$  such that  $h^{m,n} \uparrow 1_{K_n^c}$ . Therefore, if  $\phi : C_b \rightarrow \mathbb{R} \cup \{+\infty\}$  is an increasing convex functional with an increasing convex extension  $\psi$  satisfying (1.6), then for any  $f \in I(\phi)$  and  $\mu \in ca^+(\mathcal{F})$  maximizing (1.7), there exists a  $\delta > 0$  and a sequence  $(K_n)$  of compact sets such that

$$\delta \langle 1_{K_n^c}, \mu \rangle = \lim_m \delta \langle h^{m,n}, \mu \rangle \leq \lim_m \phi(f + \delta h^{m,n}) - \phi(f) \leq \psi(f + \delta 1_{K_n^c}) - \psi(f) \downarrow 0.$$

So it follows from Proposition 1.5 that  $\mu$  is regular, and as a result, the representations (1.7) and (1.9) can be written as maxima over regular finite measures or regular probability measures on  $\mathcal{F}$ , respectively.

## 2. (Adapted spaces and cones)

Let  $\psi : V \rightarrow \mathbb{R} \cup \{+\infty\}$  be an increasing convex functional, where  $V$  is an adapted space [5] or an adapted cone [10] of continuous functions  $f : \Omega \rightarrow \mathbb{R}$  on a topological space  $\Omega$ . That is,  $V$  is either a linear space satisfying

- (i)  $V = V^+ - V^+$  (where  $V^+ = \{f \in V : f \geq 0\}$ )
- (ii) For every  $\omega \in \Omega$  there exists an  $f \in V^+$  such that  $f(\omega) > 0$
- (iii) For every  $f \in V^+$ , there exists a  $g \in V^+$  such that for each  $\varepsilon > 0$  the set  $\{f > \varepsilon g\}$  is relatively compact,

or  $V$  is a convex cone with the properties

- (i)  $V = V^+ \cup \{0\}$

- (ii) For every  $\omega \in \Omega$  there exists an  $f \in V$  such that  $f(\omega) > 0$
- (iii) For every  $f \in V$ , there exists a  $g \in V$  such that for each  $\varepsilon > 0$  the set  $\{f > \varepsilon g\}$  is relatively compact.

In both cases,

$$X := \{f : \Omega \rightarrow \mathbb{R} \text{ continuous} : |f| \leq g \text{ for some } g \in V\}$$

is a Stone vector lattice containing  $V$ , and

$$\phi(f) := \inf \{\psi(g) : f \leq g, g \in V\}$$

defines an increasing convex extension  $\phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  of  $\psi$ . Furthermore, for  $f \in I(\phi)$  and a sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$ , there is an  $\varepsilon > 0$  such that  $\phi(f + \varepsilon f^1) < +\infty$ . It follows from (iii) that there exists a  $g \in V^+$  such that  $\phi(f + g) < +\infty$  and the set  $\{f^1 > g/n\}$  is relatively compact for all  $n \in \mathbb{N}$ . By compactness, one obtains from (ii) that there exist functions  $g^n \in V^+$  and numbers  $m^n \in \mathbb{N}$ ,  $n \in \mathbb{N}$ , such that  $\phi(f + g^n) < +\infty$  and  $m^n g^n \geq 1$  on  $\{f^1 > g/n\}$ . This shows that condition (B) holds with  $\hat{\phi}(h) = \phi(f + h) - \phi(f)$ . So by Theorem 1.3,

$$\phi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu)), \quad \text{for all } f \in I(\phi). \quad (1.10)$$

Moreover, it follows from the definition of  $\phi$  that  $I(\psi) \subseteq I(\phi)$  and  $\psi_V^*(\mu) = \phi_X^*(\mu)$  for  $\mu \in ca^+(X)$ . Therefore

$$\psi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \psi_V^*(\mu)), \quad \text{for all } f \in I(\psi). \quad (1.11)$$

(1.10) and (1.11) are non-linear versions of the linear representation results, Proposition 2 in [5] and Proposition 11 of [10]. But in contrast to [5, 10], here  $X$  does not have to be locally compact. As a special case of (1.11), one also recovers e.g. the max-representation of sublinear distributions given in Lemma 3.4 of [11].

The next result gives a sup-representation for increasing convex functionals  $\phi$  on the space  $B_b$  of all bounded measurable functions  $f : \Omega \rightarrow \mathbb{R}$  on a Hausdorff space  $\Omega$  with Borel  $\sigma$ -algebra  $\mathcal{F}$ . The following two conditions are variants of (vi) in Proposition 1.1. We call a sequence  $(K_n)$  of subsets of  $\Omega$  or a sequence  $(f^n)$  of real-valued functions on  $\Omega$  increasing if  $K_n \subseteq K_{n+1}$  or  $f^n \leq f^{n+1}$  for all  $n$ , respectively.

- (C)  $\phi$  is real-valued and there exists an increasing sequence  $(K_n)$  of compact subsets of  $\Omega$  such that  $\phi(f^n) \uparrow \phi(f)$  for every increasing sequence  $(f^n)$  in  $B_b$  and  $f \in B_b$  such that  $|f - f^n|_{1_{K_m}} = 0$  for all  $n \geq m$ .
- (D) There exists an increasing sequence  $(K_n)$  of compact subsets of  $\Omega$  such that  $\phi(f^n) \uparrow \phi(f)$  for every increasing sequence  $(f^n)$  in  $B_b$  and  $f \in B_b$  such that  $|f - f^n|_{1_{K_m}} \leq 1/m$  for all  $n \geq m$ .

By  $C_b$  we denote the set of all bounded continuous functions  $f : \Omega \rightarrow \mathbb{R}$  and by  $U_b$  all bounded upper semicontinuous functions  $f : \Omega \rightarrow \mathbb{R}$ . We define the lower regularization of  $\phi$  by

$$\phi_r(f) := \sup \{\phi(g) : g \in U_b, g \leq f\},$$

and say  $\phi$  is lower regular if  $\phi = \phi_r$ .  $ca_r^+(\mathcal{F})$  is the collection of all regular finite measures on  $\mathcal{F}$ . For

$$\phi_{C_b}^*(\mu) := \sup_{f \in C_b} (\langle f, \mu \rangle - \phi(f)) \quad \text{and} \quad \phi_{U_b}^*(\mu) := \sup_{f \in U_b} (\langle f, \mu \rangle - \phi(f)),$$

one obviously has  $\phi_{C_b}^*(\mu) \leq \phi_{U_b}^*(\mu)$ ,  $\mu \in ca_r^+(\mathcal{F})$ .

**Theorem 1.7** *Let  $\Omega$  be a Hausdorff space with Borel  $\sigma$ -algebra  $\mathcal{F}$  and  $\phi : B_b \rightarrow \mathbb{R} \cup \{+\infty\}$  an increasing convex functional. If  $\phi$  satisfies (C) or (D), then*

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)) \quad \text{for all } f \in C_b, \quad (1.12)$$

$$\phi(f) \leq \sup_{\mu \in ca_r^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)) \quad \text{for all } f \in U_b, \quad (1.13)$$

$$\phi_r(f) \leq \sup_{\mu \in ca_r^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)) \quad \text{for all } f \in B_b, \quad (1.14)$$

and both inequalities become equalities if  $\phi_{C_b}^*(\mu) = \phi_{U_b}^*(\mu)$  for all  $\mu \in ca_r^+(\mathcal{F})$ .

In particular, if  $\phi$  is lower regular and  $\phi_{C_b}^*(\mu) = \phi_{U_b}^*(\mu)$  for all  $\mu \in ca_r^+(\mathcal{F})$ , then

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)) \quad \text{for all } f \in B_b. \quad (1.15)$$

For positive linear functionals, Theorem 1.7 yields the following:

**Corollary 1.8** *Let  $\Omega$  be a Hausdorff space with Borel  $\sigma$ -algebra  $\mathcal{F}$  and  $\phi : B_b \rightarrow \mathbb{R}$  a positive linear functional satisfying (C). Then there exists a  $\mu \in ca_r^+(\mathcal{F})$  such that*

$$\phi(f) = \langle f, \mu \rangle \quad \text{for all } f \in C_b. \quad (1.16)$$

If  $\Omega$  is a metric space with Borel  $\sigma$ -algebra  $\mathcal{F}$ , one also has

$$\phi(f) \leq \langle f, \mu \rangle \quad \text{for all } f \in U_b \quad \text{and} \quad \phi_r(f) \leq \langle f, \mu \rangle \quad \text{for all } f \in B_b, \quad (1.17)$$

and the inequalities are equalities if  $\phi_{C_b}^*(\nu) = \phi_{U_b}^*(\nu)$  for all  $\nu \in ca_r^+(\mathcal{F})$ .

In particular, if  $\Omega$  is a metric space with Borel  $\sigma$ -algebra  $\mathcal{F}$ ,  $\phi$  is lower regular, and  $\phi_{C_b}^*(\nu) = \phi_{U_b}^*(\nu)$  for all  $\nu \in ca_r^+(\mathcal{F})$ , then

$$\phi(f) = \langle f, \mu \rangle \quad \text{for all } f \in B_b. \quad (1.18)$$

### Remarks 1.9

**1.** To have a representation of the form (1.15) or (1.18), it is necessary that  $\phi$  be lower regular. Indeed, for every  $f \in B_b$  and  $\delta > 0$ , there exists a measurable partition  $(A_m)$  of  $\Omega$  and numbers  $a_1 < \dots < a_M$  such that the step function  $g = \sum_{m=1}^M a_m 1_{A_m}$  satisfies  $g \leq f \leq g + \delta$ . Furthermore, for each  $\mu \in ca_r^+(\mathcal{F})$ , one can choose closed sets  $F_m \subseteq A_m$  such that  $\langle g, \mu \rangle \leq \langle h, \mu \rangle + \delta$  for the upper semicontinuous function  $h = a_1 1_{\Omega \setminus \bigcup_{m=2}^M F_m} + \sum_{m=2}^M a_m 1_{F_m} \leq g$ . It follows that  $\langle f, \mu \rangle \leq \langle h, \mu \rangle + \delta(\langle 1, \mu \rangle + 1)$ . So any linear functional of the form (1.18) is lower regular, and as a supremum of lower regular functionals, (1.15) is

again lower regular.

**2.** If  $\Omega$  is a Hausdorff space with Borel  $\sigma$ -algebra  $\mathcal{F}$ , it follows from 1. that for all  $\mu \in ca_r^+(\mathcal{F})$  and  $f \in B_b$ , there exists a sequence  $(f^n)$  in  $U_b$  such that  $f^n \leq f$  and  $\langle f^n, \mu \rangle \uparrow \langle f, \mu \rangle$ . As a result, one obtains for every increasing functional  $\phi : B_b \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\mu \in ca_r^+(\mathcal{F})$ ,

$$\phi_{U_b}^*(\mu) = \sup_{f \in U_b} (\langle f, \mu \rangle - \phi(f)) = \phi_{B_b}^*(\mu) = \sup_{f \in B_b} (\langle f, \mu \rangle - \phi(f)).$$

Similarly, if  $\mu \in ca_r^+(\mathcal{F})$  has the property that for all  $f \in U_b$ , there exists a sequence  $(f^n)$  in  $C_b$  such that  $f^n \leq f$  and  $\langle f^n, \mu \rangle \uparrow \langle f, \mu \rangle$ , then  $\phi_{C_b}^*(\mu) = \phi_{U_b}^*(\mu)$  for every increasing functional  $\phi : B_b \rightarrow \mathbb{R} \cup \{+\infty\}$ . This provides a sufficient condition for the inequalities in (1.13), (1.14) and (1.17) to be equalities.

The remainder of the paper is organized as follows: In Section 2 we prove representation (1.1), Proposition 1.1, Theorem 1.3, Corollary 1.4 and Proposition 1.5. In Section 3 we give the proofs of Theorem 1.7 and Corollary 1.8.

## 2 Derivation of max-representations

### Proof of the representation (1.1)

It is immediate from the definition of  $\phi_X^*$  that

$$\phi(f) \geq \sup_{\mu \in ba^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_X^*(\mu)) \quad \text{for every } f \in X. \quad (2.1)$$

On the other hand, for  $f \in I(\phi)$ , the directional derivative

$$\phi'(f; g) := \lim_{\varepsilon \downarrow 0} \frac{\phi(f + \varepsilon g) - \phi(f)}{\varepsilon}$$

is a real-valued increasing sublinear function of  $g \in X$ . So it follows from the Hahn–Banach extension theorem that there exists a positive linear functional  $\psi : X \rightarrow \mathbb{R}$  satisfying

$$\psi(g) \leq \phi'(f; g) \leq \phi(f + g) - \phi(f)$$

for all  $g \in X$ . Since  $\psi(\lambda 1) = \lambda \psi(1)$ ,  $\lambda \in \mathbb{R}$ , one obtains by monotonicity that  $\psi$  is continuous with respect to the sup-norm on  $X$ . Therefore, it can be represented as  $\psi(g) = \langle g, \nu \rangle$  for the finitely additive measure  $\nu \in ba^+(\mathcal{F})$  given by  $\nu(A) := \psi(1_A)$ ,  $A \in \mathcal{F}$ . It follows that  $\phi(f) + \phi_X^*(\nu) = \langle f, \nu \rangle$ , which together with (2.1), implies  $\phi(f) = \max_{\mu \in ba^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_X^*(\mu))$ .  $\square$

### Proof of Proposition 1.1

To prove (i)  $\Rightarrow$  (ii), let  $f, g \in I(\phi)$  such that  $f$  fulfills (i). Then for all  $\lambda \in (0, 1)$  and every sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$ , one has

$$\begin{aligned} \phi(g + f^n) &\leq \lambda \phi\left(f + \frac{1}{\lambda} f^n\right) + (1 - \lambda) \phi\left(\frac{g - \lambda f}{1 - \lambda}\right) \\ &= \lambda \phi\left(f + \frac{1}{\lambda} f^n\right) + (1 - \lambda) \phi\left(g + \frac{\lambda}{1 - \lambda} (g - f)\right). \end{aligned}$$



Since  $f$  satisfies (i),

$$\phi\left(f + \frac{1}{\lambda}f^n\right) \downarrow \phi(f) \quad \text{for fixed } \lambda \in (0, 1) \text{ and } n \rightarrow \infty.$$

Moreover, there exists a  $\delta > 0$  such that  $x \mapsto \phi(g + x(g - f))$  is a real-valued convex function on the interval  $(-\delta, \delta)$ . As a consequence, it is continuous at 0, and one obtains

$$\lambda\phi(f) + (1 - \lambda)\phi\left(g + \frac{\lambda}{1 - \lambda}(g - f)\right) \rightarrow \phi(g) \quad \text{for } \lambda \downarrow 0.$$

This shows that  $\phi(g + f^n) \downarrow \phi(g)$ .

(ii)  $\Rightarrow$  (iii) is obvious. To prove (iii)  $\Rightarrow$  (iv), note first that it follows from the definition of  $\phi_X^*$  that

$$\phi(f) \geq \sup_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu)) \quad \text{for all } f \in X.$$

Moreover, for  $f \in I(\phi)$ , one deduces as in the proof of the representation (1.1) that there exists a positive linear functional  $\psi : X \rightarrow \mathbb{R}$  satisfying

$$\psi(g) \leq \phi'(f; g) \leq \phi(f + g) - \phi(f), \quad g \in X.$$

If (iii) holds, then for every sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$ , there exists an  $\varepsilon > 0$  such that

$$\varepsilon\psi(f^n) \leq \phi(f + \varepsilon f^n) - \phi(f) \downarrow 0.$$

So one obtains from the Daniell–Stone theorem a  $\nu \in ca^+(X)$  such that  $\psi(g) = \langle g, \nu \rangle$  for all  $g \in X$ . It follows that  $\phi(f) + \phi_X^*(\nu) = \langle f, \nu \rangle$ , which implies  $\phi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu))$ .

(iv)  $\Rightarrow$  (v) is clear, and (v)  $\Rightarrow$  (vi) follows since by the monotone convergence theorem, the mapping  $f \mapsto \langle f, \mu \rangle - \phi_X^*(\mu)$  satisfies (v) for every  $\mu \in ca^+(X)$ .  $\square$

### Proof of Theorem 1.3

Choose a function  $f \in I(\phi)$  and a sequence  $(f^n)$  in  $X^+$  satisfying  $f^n \downarrow 0$ . If we can show that there exists an  $\varepsilon > 0$  such that  $\phi(f + \varepsilon f^n) \downarrow \phi(f)$ , the theorem follows from Proposition 1.1. Let us first assume  $\phi$  satisfies (A). Then there exists a  $\lambda > 0$  such that for every  $\delta > 0$ , there are  $m \in \mathbb{N}$ ,  $g \in \mathbb{R}_+^\Omega$  and an increasing convex function  $\hat{\phi} : Y \rightarrow \mathbb{R}$  on a convex subset  $Y \subseteq \mathbb{R}_+^\Omega$  containing  $\{0, f^m g, (\lambda - g)^+, \lambda f^n : n \geq m\}$  such that  $\{g < \lambda\}$  is relatively compact,  $\hat{\phi}(f^m g) \leq \delta$ , and  $\hat{\phi}(\lambda f^n) \geq \phi(f + \lambda f^n) - \phi(f)$  for all  $n \geq m$ . Since  $x \mapsto \hat{\phi}(x(\lambda - g)^+)$  is a real-valued increasing convex function on the interval  $[0, 1]$ , it must be continuous at 0. In particular, there exists an  $x \in (0, 1]$  such that

$$\hat{\phi}(x(\lambda - g)^+) \leq \hat{\phi}(0) + \delta \leq \hat{\phi}(f^m g) + \delta \leq 2\delta.$$

For  $n \geq m$ , one has  $\lambda f^n \leq f^m g + f^n(\lambda - g)^+$ , and by Dini's lemma,  $f^n$  converges to 0 uniformly on the closure of  $\{g < \lambda\}$ . So there exists an  $n \geq m$  such that

$$f^n(\lambda - g)^+ \leq x(\lambda - g)^+,$$

and one obtains

$$\phi\left(f + \frac{\lambda}{2}f^n\right) - \phi(f) \leq \hat{\phi}\left(\frac{\lambda}{2}f^n\right) \leq \hat{\phi}\left(\frac{f^m g + x(\lambda - g)^+}{2}\right) \leq \frac{\hat{\phi}(f^m g) + \hat{\phi}(x(\lambda - g)^+)}{2} \leq 2\delta.$$

Since  $\delta > 0$  was arbitrary, this shows that  $\phi(f + \lambda f^n/2) \downarrow \phi(f)$ .

If instead of (A),  $f$  satisfies condition (B), there exist functions  $g, g^1, g^2, \dots$  in  $\mathbb{R}_+^\Omega$  and numbers  $m, m^1, m^2, \dots$  in  $\mathbb{N}$  together with an increasing convex function  $\hat{\phi} : Y \rightarrow \mathbb{R}$  on a convex subset  $Y \subseteq \mathbb{R}_+^\Omega$  containing  $\{0, f^n/m, g, g^n : n \geq m\}$  such that  $\{f^m > g/n\}$  is relatively compact and contained in  $\{m^n g^n \geq 1\}$  for all  $n \geq m$ ,  $\hat{\phi}(0) = 0$ , and  $\hat{\phi}(f^n/m) \leq \phi(f + f^n/m) - \phi(f)$  for all  $n \geq m$ . Since  $x \mapsto \hat{\phi}(xg)$  is a real-valued increasing convex function on the interval  $[0, 1]$ , it is continuous at 0. In particular, for given  $\delta > 0$ , there exists an integer  $k \geq 2m$  such that  $\hat{\phi}(2g/km) \leq \delta$ . Similarly, there exists an integer  $l \geq 2m^k$  such that  $\hat{\phi}(2m^k g^k/lm) \leq \delta$ . By Dini's Lemma,  $f^n$  converges uniformly to 0 on the closure of the set  $\{f^m > g/k\}$ . So there exists an  $n \geq m$  such that  $f^n \leq 1/l$  on  $\{f^m > g/k\}$ . Since  $\{f^m > g/k\}$  is contained in  $\{m^k g^k \geq 1\}$  and  $f^n \leq f^m \leq g/k$  on  $\{f^m \leq g/k\}$ , one has  $(f^n - g/k)^+ \leq m^k g^k/l$ . Therefore,

$$f^n \leq g/k + (f^n - g/k)^+ \leq g/k + m^k g^k/l,$$

and

$$\phi\left(f + \frac{f^n}{m}\right) - \phi(f) \leq \hat{\phi}\left(\frac{f^n}{m}\right) \leq \hat{\phi}\left(\frac{g}{km} + \frac{m^k g^k}{lm}\right) \leq \frac{\hat{\phi}(2g/km) + \hat{\phi}(2m^k g^k/lm)}{2} \leq \delta.$$

This shows that  $\phi(f + f^n/m) \downarrow \phi(f)$ , and the proof is complete.  $\square$

#### Proof of Corollary 1.4

It follows from Theorem 1.3 that there exists a  $\mu \in ca^+(X)$  such that  $\phi_X^*(\mu) < +\infty$ . If  $\phi$  is linear, this implies that  $\langle f, \mu \rangle = \phi(f)$  for all  $f \in X$ .  $\square$

#### Proof of Proposition 1.5

Fix a finite measure  $\mu$  on  $\sigma(X)$  and call a set  $A \in \sigma(X)$  closed regular if

$$\mu(A) = \sup\{\mu(B) : B \in \sigma(X), B \text{ is closed and } B \subseteq A\}.$$

The collection of sets

$$\mathcal{G} := \{A \in \sigma(X) : A \text{ and } \Omega \setminus A \text{ are closed regular}\}$$

forms a sub- $\sigma$ -algebra of  $\sigma(X)$ . For a closed set  $F \subseteq \mathbb{R}$  and  $f \in X$ ,  $f^{-1}(F)$  is a closed subset of  $\Omega$ . Moreover,  $\mathbb{R} \setminus F$  can be written as a countable union  $\bigcup_n F_n$  of closed sets  $F_n \subseteq \mathbb{R}$ . Therefore,  $\Omega \setminus f^{-1}(F)$  equals  $\bigcup_n f^{-1}(F_n)$ , which can be approximated with the closed sets  $\bigcup_{n=1}^N f^{-1}(F_n)$ . This shows that  $f^{-1}(F)$  belongs to  $\mathcal{G}$ . Since the sets  $f^{-1}(F)$  generate  $\sigma(X)$ , one obtains  $\mathcal{G} = \sigma(X)$ , which means that  $\mu$  is closed regular.

If there exists a sequence  $(K_n)$  of compact sets in  $\sigma(X)$  such that  $\mu(K_n) \rightarrow \mu(\Omega)$ , then  $\mu(A \cap K_n) \rightarrow \mu(A)$  for every  $A \in \mathcal{F}$ . Moreover, for every  $n$  there exists a closed set  $B_n \subseteq A \cap K_n$  in  $\sigma(X)$  such that  $\mu(B_n) \geq \mu(A \cap K_n) - 1/n$ . Since every closed subset of a compact set is compact, this shows that  $\mu$  is regular.  $\square$

### 3 Derivation of sup-representations

For a sequence of compact Hausdorff spaces  $(H_n)$ , consider the sequence spaces

$$U := \left\{ u \in \prod_n C(H_n) : \|u\| < \infty \right\} \quad \text{and} \quad V := \left\{ \nu \in \prod_n ca_r(H_n) : \|\nu\| < \infty \right\},$$

where  $C(H_n)$  denotes the set of all real-valued continuous functions on  $H_n$ ,  $ca_r(H_n) = ca_r^+(H_n) - ca_r^+(H_n)$ , where  $ca_r^+(H_n)$  are all finite regular measures on the Borel  $\sigma$ -algebra of  $H_n$ , and the norms are defined as follows:

$$\begin{aligned} \|u\| &:= \sup_n \|u_n\|_\infty \text{ for the sup-norm } \|\cdot\|_\infty \text{ and} \\ \|\nu\| &:= \sum_n \|\nu_n\|_{\text{tv}} < \infty \text{ for the total variation norm } \|\cdot\|_{\text{tv}}. \end{aligned}$$

By the Riesz representation theorem (see e.g. Theorem IV.6.3 in [8]),  $ca_r(H_n)$  is the topological dual of  $C(H_n)$ . Therefore,  $(U, V)$  is a dual pair under the bilinear form  $\langle u, \nu \rangle := \sum_n \langle u_n, \nu_n \rangle$ . By  $V^+$  we denote the set of all  $\nu \in V$  belonging to  $\prod_n ca_r^+(H_n)$ . For a function  $\psi : U \rightarrow \mathbb{R} \cup \{+\infty\}$ , we consider the following two conditions:

- (C')  $\psi$  is real-valued and  $\lim_n \psi(u^n) = \psi(u)$  for every increasing sequence  $(u^n)$  in  $U$  and  $u \in U$  such that  $u_m^n = u_m$  for all  $n \geq m$ .
- (D')  $\lim_n \psi(u^n) = \psi(u)$  for every increasing sequence  $(u^n)$  in  $U$  and  $u \in U$  such that  $\lim_n \|u_m^n - u_m\|_\infty = 0$  for every  $m$ .

Note that  $U$  contains  $l^\infty$  as a subspace, and on  $l^\infty$  the following holds:

**Lemma 3.1** *Every increasing convex functional  $\psi : l^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfying (C') or (D') is  $\sigma(l^\infty, l^1)$ -lower semicontinuous.*

*Proof.* To prove the lemma one has to show that all lower level sets of  $\psi$  are  $\sigma(l^\infty, l^1)$ -closed. By the Krein-Šmulian theorem (see e.g. Theorem V.5.7 in [8]), it is enough to show that the sets

$$D_{a,b} = \{x \in l^\infty : \psi(x) \leq a, \|x\|_\infty \leq b\}, \quad a, b \in \mathbb{R},$$

are  $\sigma(l^\infty, l^1)$ -closed, which they are if and only if they are  $\sigma(l^\infty, l^1(\eta))$ -closed, where  $l^1(\eta)$  is the  $l^1$ -space with respect to the probability measure  $\eta$  on  $\mathbb{N}$  given by  $\eta(n) = 2^{-n}$ , and the pairing on  $(l^\infty, l^1(\eta))$  is  $\langle x, y \rangle = \sum_n x_n y_n 2^{-n}$ . The embedding of  $l^\infty$  in  $l^1(\eta)$  is continuous with respect to  $\sigma(l^\infty, l^1(\eta))$  and  $\sigma(l^1(\eta), l^\infty)$ . So it is sufficient to show that the sets  $D_{a,b}$  are  $\sigma(l^1(\eta), l^\infty)$ -closed. But by convexity, this follows if it can be shown that they are norm-closed in  $l^1(\eta)$ . To do that, consider a sequence  $(x^n)$  in  $D_{a,b}$  converging to  $x$  in the  $l^1(\eta)$ -norm. Then  $\|x\|_\infty \leq b$ , and  $y_m^n := \inf_{j \geq n} x_m^j$  defines a sequence  $(y^n)$  in  $D_{a,b}$  which increases component-wise to  $x$ .

Under (C')  $\psi$  is real-valued, and since every real-valued convex function on  $\mathbb{R}^m$  is continuous, one has

$$\psi(x_1, \dots, x_m, y_{m+1}^1, y_{m+2}^1, \dots) = \lim_n \psi(y_1^n, \dots, y_m^n, y_{m+1}^1, y_{m+2}^1, \dots) \leq \lim_n \psi(y^n) \leq a$$

for all  $m \geq 1$ . Therefore, it follows from (C') that  $x$  belongs to  $D_{a,b}$ . If  $\psi$  satisfies (D'), one obtains  $\psi(x) = \lim_n \psi(y^n) \leq a$ . So  $x$  belongs to  $D_{a,b}$ .  $\square$

We also need the following

**Lemma 3.2** *For every  $y \in l^1$ , the set  $A_y = \{\nu \in V : \|\nu_n\|_{\text{tv}} \leq |y_n|\}$  is  $\sigma(V, U)$ -compact.*

*Proof.*

$$\tilde{U} := \left\{ u \in \prod_n C(H_n) : \sum_n \|u_n\|_\infty < \infty \right\}$$

is a Banach space with topological dual

$$\tilde{V} := \left\{ \nu \in \prod_n ca_r(H_n) : \sup_n \|\nu_n\|_{\text{tv}} < \infty \right\}.$$

Therefore, one obtains from the Banach–Alaoglu theorem that the norm ball  $\{\nu \in \tilde{V} : \sup_n \|\nu_n\|_{\text{tv}} \leq 1\}$  is  $\sigma(\tilde{V}, \tilde{U})$ -compact. But for  $y \in l^1$ , the mapping  $(\nu_n) \mapsto (\nu_n y_n)$  continuously embeds  $\tilde{V}$  in  $V$  with respect to  $\sigma(\tilde{V}, \tilde{U})$  and  $\sigma(V, U)$ . It follows that  $A_y$  is  $\sigma(V, U)$ -compact.  $\square$

Now we are ready to prove a representation result for increasing convex functionals on  $U$ .

**Proposition 3.3** *Every increasing convex functional  $\psi : U \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfying (C') or (D') has a representation of the form*

$$\psi(u) = \sup_{\nu \in V^+} (\langle u, \nu \rangle - \psi^*(\nu)) \quad \text{for} \quad \psi^*(\nu) := \sup_{u \in U} (\langle u, \nu \rangle - \psi(u)).$$

*Proof.* In the case  $\psi \equiv +\infty$ , the proposition is clear. So let us assume that  $\psi(u) < +\infty$  for at least one  $u \in U$ . Then it is enough to show that

$$\psi(u) = \sup_{\nu \in V} (\langle u, \nu \rangle - \psi^*(\nu)), \quad u \in U, \tag{3.1}$$

since it follows from the monotonicity of  $\psi$  that  $\psi^*(\nu) = +\infty$  for all  $\nu \in V \setminus V^+$ . But (3.1) is a consequence of the Fenchel–Moreau theorem (see e.g. Theorem 3.2.2 in [13]) if we can show that  $\psi$  is  $\sigma(U, V)$ -lower semicontinuous, or equivalently, all lower level sets  $D_a = \{u \in U : \psi(u) \leq a\}$  are  $\sigma(U, V)$ -closed. Moreover, since every  $D_a$  is convex, it follows from the Hahn–Banach separation theorem together with the Mackey–Arens theorem (see e.g. Theorem IV.3.2 in [12]) that it is  $\sigma(U, V)$ -closed if we can show that it is closed in the Mackey topology  $\tau(U, V)$ . So let  $(u^\alpha)$  be a net in  $D_a$  such that  $u^\alpha \rightarrow \hat{u} \in U$  in  $\tau(U, V)$ . We know from Lemma 3.2 that for every  $y \in l^1$ , the set

$$A_y := \{\nu \in V : \|\nu_n\|_{\text{tv}} \leq |y_n|\}$$

is  $\sigma(V, U)$ -compact. Therefore, one has

$$\sum_n \|u_n^\alpha - \hat{u}_n\|_\infty |y_n| \leq \sup_{\nu \in A_y} |\langle u^\alpha - \hat{u}, \nu \rangle| \rightarrow 0. \quad (3.2)$$

If  $\psi$  satisfies (C'), we define the projections  $\pi_n : U \rightarrow l^\infty$  as follows: for  $m > n$ ,

$$\pi_n(u)_m := \underline{u}_m := \min_{z \in H_m} u_m(z),$$

and for  $m = 1, \dots, n$ ,

$$\begin{aligned} \pi_n(u)_1 &:= \min \{x \in \mathbb{R} : x \geq \underline{u}_1, \psi(x, u_2, \dots, u_n, \underline{u}_{n+1}, \dots) = \psi(u_1, u_2, \dots, u_n, \underline{u}_{n+1}, \dots)\} \\ \pi_n(u)_2 &:= \min \{x \in \mathbb{R} : x \geq \underline{u}_2, \psi(\pi_n(u)_1, x, u_3, \dots, u_n, \underline{u}_{n+1}, \dots) = \psi(u_1, \dots, u_n, \underline{u}_{n+1}, \dots)\} \\ &\dots \\ \pi_n(u)_n &:= \min \{x \in \mathbb{R} : x \geq \underline{u}_n, \psi(\pi_n(u)_1, \dots, \pi_n(u)_{n-1}, x, \underline{u}_{n+1}, \dots) = \psi(u_1, \dots, u_n, \underline{u}_{n+1}, \dots)\}. \end{aligned}$$

Since  $x \mapsto \psi(x, u_2, \dots, u_n, \underline{u}_{n+1}, \dots)$  is a convex function from  $\mathbb{R}$  to  $\mathbb{R}$ , it is continuous. Therefore, the minimum in the definition of  $\pi_n(u)_1$  is attained, and

$$\psi(\pi_n(u)_1, u_2, \dots, u_n, \underline{u}_{n+1}, \dots) = \psi(u_1, u_2, \dots, u_n, \underline{u}_{n+1}, \dots).$$

Analogously, the other minima are attained, and

$$\psi \circ \pi_n(u) = \psi(u_1, \dots, u_n, \underline{u}_{n+1}, \dots) \quad \text{for all } u \in U.$$

Since  $\psi$  is increasing,  $\pi_n(u^\alpha)$  is in  $D_a$  for all  $\alpha$ , and by (3.2), one has for each  $y \in l^1$ ,

$$|\langle \pi_n(u^\alpha) - \pi_n(\hat{u}), y \rangle| \leq \sum_m \|u_m^\alpha - \hat{u}_m\|_\infty |y_m| \rightarrow 0,$$

showing that  $\pi_n(u^\alpha) \rightarrow \pi_n(\hat{u})$  in  $\sigma(l^\infty, l^1)$ . From Lemma 3.1 we know that  $\psi$  restricted to  $l^\infty$  is  $\sigma(l^\infty, l^1)$ -lower semicontinuous. Therefore,  $\pi_n(\hat{u})$  is in  $D_a$  for all  $n$ , and one obtains from (C') that

$$\psi \circ \pi_n(\hat{u}) = \psi(\hat{u}_1, \dots, \hat{u}_n, \hat{u}_{n+1}, \dots) \uparrow \psi(\hat{u}) \quad \text{for } n \rightarrow \infty.$$

This shows that  $\hat{u}$  belongs to  $D_a$ , which completes the proof in the case where  $\psi$  satisfies (C').

If  $\psi$  fulfills (D'), we fix  $n \geq 1$  and note that due to (3.2), there exists an  $\alpha_0$  such that

$$u_m^\alpha \geq \hat{u}_m - \frac{1}{n} \quad \text{for all } \alpha \geq \alpha_0 \text{ and } m = 1, \dots, n.$$

It follows that

$$\left( \hat{u}_1 - \frac{1}{n}, \dots, \hat{u}_n - \frac{1}{n}, u_{n+1}^\alpha - \frac{1}{n}, \dots \right) \text{ is in } D_a \text{ for all } \alpha \geq \alpha_0.$$

As above, one deduces from (3.2) that

$$\left( u_{n+1}^\alpha - \frac{1}{n}, u_{n+2}^\alpha - \frac{1}{n}, \dots \right) \rightarrow \left( \hat{u}_{n+1} - \frac{1}{n}, \hat{u}_{n+2} - \frac{1}{n}, \dots \right) \text{ in } \sigma(l^\infty, l^1).$$

So, since

$$x \mapsto \psi \left( \hat{u}_1 - \frac{1}{n}, \dots, \hat{u}_n - \frac{1}{n}, x_1, x_2, \dots \right)$$

defines an increasing convex mapping on  $l^\infty$  with property (D'), one obtains from Lemma 3.1 that

$$\left( \hat{u}_1 - \frac{1}{n}, \dots, \hat{u}_n - \frac{1}{n}, \hat{u}_{n+1} - \frac{1}{n}, \hat{u}_{n+2} - \frac{1}{n}, \dots \right) \text{ belongs to } D_a \text{ for all } n \geq 1.$$

Now it follows from (D') that  $\hat{u}$  is in  $D_a$ , and the proof is complete.  $\square$

### Proof of Theorem 1.7

We first prove (1.12). It is immediate from the definition of  $\phi_{C_b}^*$  that  $\phi(f) \geq \sup_{\mu \in ca_r^+} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu))$  for all  $f \in C_b$ . We show the other inequality in the following three steps:

Step 1: For  $H_n = K_n$ , define the function  $\psi : U = \prod_n C(H_n) \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$\psi(u) := \phi \left( \sum_n u_n 1_{K_n \setminus K_{n-1}} \right), \quad \text{where } K_0 := \emptyset.$$

Then  $\psi$  is increasing and convex. Moreover, it fulfills (C') or (D') if  $\phi$  satisfies (C) or (D), respectively. So it follows from Proposition 3.3 that  $\psi$  can be represented as

$$\psi(u) = \sup_{\nu \in V^+} (\langle u, \nu \rangle - \psi^*(\nu)).$$

Step 2: For every  $\nu \in V^+$ ,  $\mu_\nu(A) = \sum_n \nu_n(A \cap K_n)$  defines an element of  $ca_r^+(\mathcal{F})$ . Indeed,  $\mu_\nu$  is a finite measure since  $\|\nu\| = \sum_n \|\nu_n\|_{\text{tv}} < \infty$ . Moreover, for given  $A \in \mathcal{F}$  and  $\varepsilon > 0$ , there exist compact sets  $F_n \subseteq A \cap K_n$  such that  $\nu_n(F_n) \geq \nu_n(A \cap K_n) - 2^{-n-1}\varepsilon$ . So for  $m \in \mathbb{N}$  large enough,  $F = \bigcup_{n=1}^m F_n$  is compact,  $F \subseteq A$  and  $\mu_\nu(F) \geq \mu_\nu(A) - \varepsilon$ .

Step 3: Since  $\phi$  satisfies (C) or (D), one has for each  $f \in C_b$ ,

$$\phi(f) = \phi(f 1_{\bigcup_n K_n}) = \psi(f|_{K_1}, f|_{K_2}, \dots).$$

Therefore,

$$\phi(f) = \sup_{\nu \in V^+} \left( \sum_n \langle f|_{K_n}, \nu_n \rangle - \psi^*(\nu) \right) = \sup_{\nu \in V^+} (\langle f, \mu_\nu \rangle - \psi^*(\nu)),$$

and it is enough to show that  $\phi_{C_b}^*(\mu_\nu) \leq \psi^*(\nu)$  for all  $\nu \in V^+$  to complete the proof of (1.12). But this readily follows from

$$\begin{aligned} \phi_{C_b}^*(\mu_\nu) &= \sup_{f \in C_b} (\langle f, \mu_\nu \rangle - \phi(f)) = \sup_{f \in C_b} \left( \sum_n \langle f 1_{K_n}, \nu_n \rangle - \psi(f|_{K_1}, f|_{K_2}, \dots) \right) \\ &\leq \sup_{u \in U} (\langle u, \nu \rangle - \psi(u)) = \psi^*(\nu). \end{aligned}$$

To show (1.13) we fix an  $f \in U_b$  and a constant  $\varepsilon > 0$ . For every  $\delta > 0$ , there exists a measurable partition  $(A_m)$  of  $\Omega$  and real numbers  $a_1 < \dots < a_M$  such that the step function  $g = \sum_{m=1}^M a_m 1_{A_m}$

satisfies  $g \leq f \leq g + \delta$ , and by passing to the upper semicontinuous hull, one can assume  $g$  to be upper semicontinuous. If  $\phi$  satisfies (C), then  $x \mapsto \phi(f + x)$  defines a convex function from  $\mathbb{R}$  to  $\mathbb{R}$ . So it has to be continuous, and since  $\phi$  is increasing, one can ensure that  $\phi(g) \geq \phi(f) - \varepsilon$  by choosing  $\delta > 0$  small enough. If  $\phi$  satisfies (D) and  $\phi(f) < +\infty$ , one obtains directly that  $\phi(g) \geq \phi(f) - \varepsilon$  for  $\delta > 0$  small enough. On the other hand, if  $\phi$  satisfies (D) and  $\phi(f) = +\infty$ , then  $\phi(g) \geq \varepsilon$  for  $\delta > 0$  small enough. Now denote

$$U^M := \left\{ u \in \prod_n C(K_n)^M : \sup_{n,m} \|u_{nm}\|_\infty < \infty \right\}, \quad V^M := \left\{ \nu \in \prod_n ca_r(K_n)^M : \sum_{n,m} \|\nu_{nm}\|_{\text{tv}} < \infty \right\},$$

and define  $\psi : U^M \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$\psi(u) := \phi \left( \sum_n \sum_{m=1}^M u_{nm} 1_{B_{nm}} \right), \quad \text{where } K_0 := \emptyset \text{ and } B_{nm} := (K_n \setminus K_{n-1}) \cap A_m.$$

Then  $\psi$  is increasing, convex and satisfies (C') or (D'). Therefore, it follows from Proposition 3.3 that

$$\psi(u) = \sup_{\nu \in (V^M)^+} (\langle u, \nu \rangle - \psi^*(\nu)), \quad \text{where } \psi^*(\nu) = \sup_{u \in U^M} (\langle u, \nu \rangle - \psi(u)).$$

If  $\phi(h) = +\infty$  for all  $h \in C_b$ , then  $\phi_{C_b}^* \equiv -\infty$ , and (1.13) is obvious. So let us assume there exists an  $h \in C_b$  such that  $\phi(h) < +\infty$ . Then it follows that  $\nu_{nm}(K_n \setminus \bar{B}_{nm}) = 0$  for all  $\nu \in (V^M)^+$  satisfying  $\psi^*(\nu) < +\infty$ . Indeed, assume  $\nu_{nm}(K_n \setminus \bar{B}_{nm}) > 0$ . Then, since  $\nu_{nm}$  is regular, there exists a closed set  $F \subseteq K_n \setminus \bar{B}_{nm}$  with  $\nu_{nm}(F) > 0$ . By Theorem 2.48 in [1],  $K_n$  is normal. So it follows from Urysohn's lemma that there exists a continuous function  $\varphi : K_n \rightarrow [0, 1]$  which is 1 on  $F$  and 0 on  $\bar{B}_{nm}$ . This gives

$$\begin{aligned} \psi^*(\nu) &\geq \sup_{x \in \mathbb{R}_+} \left( \sum_i \sum_{j=1}^M \langle h 1_{K_i}, \nu_{ij} \rangle + \langle x \varphi, \nu_{nm} \rangle - \phi(h + x \varphi 1_{B_{nm}}) \right) \\ &= \sup_{x \in \mathbb{R}_+} \left( \sum_i \sum_{j=1}^M \langle h 1_{K_i}, \nu_{ij} \rangle + \langle x \varphi, \nu_{nm} \rangle - \phi(h) \right) = +\infty. \end{aligned}$$

Now define  $u \in U^M$  by  $u_{nm} = a_m$ . It follows from (C) or (D) that

$$\phi(g) = \phi(g 1_{\bigcup_n K_n}) = \psi(u).$$

Therefore, since  $g$  is upper semicontinuous,

$$\begin{aligned} \phi(g) &= \sup_{\nu \in (V^N)^+} \left( \sum_n \sum_{m=1}^M \langle u_{nm}, \nu_{nm} \rangle - \psi^*(\nu) \right) \\ &\leq \sup_{\nu \in (V^N)^+} \left( \sum_n \sum_{m=1}^M \langle g 1_{\bar{B}_{nm}}, \nu_{nm} \rangle - \psi^*(\nu) \right) = \sup_{\nu \in (V^N)^+} (\langle g, \mu_\nu \rangle - \psi^*(\nu)), \end{aligned}$$

where  $\mu_\nu$  is given by  $\mu_\nu(A) := \sum_n \sum_{m=1}^M \nu_{nm}(A \cap K_n)$ . It follows as above that  $\mu_\nu$  belongs to  $ca_r^+(\mathcal{F})$ , and for all  $\nu \in (V^M)^+$ , one has

$$\begin{aligned} \phi_{C_b}^*(\mu_\nu) &= \sup_{l \in C_b} (\langle l, \mu_\nu \rangle - \phi(l)) \\ &= \sup_{l \in C_b} \left( \sum_n \sum_{m=1}^M \langle l 1_{K_n}, \nu_{nm} \rangle - \psi(l 1_{K_1}, \dots, l 1_{K_1}, l 1_{K_2}, \dots) \right) \\ &\leq \sup_{u \in U^M} (\langle u, \nu \rangle - \psi(u)) = \psi^*(\nu). \end{aligned}$$

So in the case  $\phi(f) < +\infty$ , one obtains

$$\phi(f) - \varepsilon \leq \phi(g) \leq \sup_{\mu \in ca_r^+} (\langle g, \mu \rangle - \phi_{C_b}^*(\mu)) \leq \sup_{\mu \in ca_r^+} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)),$$

and if  $\phi(f) = +\infty$ ,

$$\varepsilon \leq \phi(g) \leq \sup_{\mu \in ca_r^+} (\langle g, \mu \rangle - \phi_{C_b}^*(\mu)) \leq \sup_{\mu \in ca_r^+} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)).$$

Since  $\varepsilon > 0$  was arbitrary, this yields (1.13). On the other hand, it follows from the definition of  $\phi_{U_b}^*$  that  $\phi(f) \geq \sup_{\mu \in ca_r^+} (\langle f, \mu \rangle - \phi_{U_b}^*(\mu))$ . So if  $\phi_{C_b}^*(\mu) = \phi_{U_b}^*(\mu)$  for all  $\mu \in ca_r^+$ , the inequality in (1.13) becomes an equality.

Finally, by Remark 1.9.1,  $\hat{\phi}(f) := \sup_{\mu \in ca_r^+} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu))$  is lower regular on  $B_b$ . So one obtains from the second part of the proof that for all  $f \in B_b$ ,

$$\phi_r(f) = \sup\{\phi(g) : g \in U_b, g \leq f\} \leq \sup\{\hat{\phi}(g) : g \in U_b, g \leq f\} = \hat{\phi}(f),$$

with equality if  $\phi_{C_b}^*(\mu) = \phi_{U_b}^*(\mu)$  for all  $\mu \in ca_r^+$ . This completes the proof.  $\square$

### Proof of Corollary 1.8

By Theorem 1.7, one has

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{F})} (\langle f, \mu \rangle - \phi_{C_b}^*(\mu)) \quad \text{for all } f \in C_b.$$

In particular,  $\phi_{C_b}^*(\mu) < +\infty$  for at least one  $\mu \in ca_r^+(\mathcal{F})$ . Since  $\phi$  is linear, this implies that  $\phi(f) = \langle f, \mu \rangle$  for all  $f \in C_b$ , and  $\phi_{C_b}^*(\mu) = 0$ . Moreover, if  $\Omega$  is a metric space,  $\mu$  is completely determined by the values  $\langle f, \mu \rangle$ ,  $f \in C_b$  (see e.g. [3]). So one obtains from (1.13) and (1.14) that  $\phi(f) \leq \langle f, \mu \rangle$  for all  $f \in U_b$  and  $\phi_r(f) \leq \langle f, \mu \rangle$  for all  $f \in B_b$ , with equality if  $\phi_{C_b}^*(\nu) = \phi_{U_b}^*(\nu)$  for all  $\nu \in ca_r^+(\mathcal{F})$ .  $\square$

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