Multidimensional Markovian FBSDEs with super-quadratic growth

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ABSTRACT

We give local and global existence and uniqueness results for multidimensional coupled FBSDEs for generators with arbitrary growth in the control variable. The local existence result is based on Malliavin calculus arguments for Markovian equations. Under additional monotonicity conditions on the generator we construct global solutions by a pasting technique along PDE solutions.

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1 Introduction

Given a multidimensional Brownian motion W on a probability space, we consider the system of forward and backward stochastic differential equations

$$\begin{cases} X_t &= x + \int_0^t b_s(X_s, Y_s)\,ds + \int_0^t \sigma_s dW_s \\ Y_t &= h(X_T) + \int_t^T g_s(X_s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s, \quad t \in [0,T] \end{cases} \tag{1.1}$$

where x is the initial value, $T > 0$ is a finite time horizon, and b, σ, g and h are given functions. In this paper, we give conditions under which the system admits a unique solution in the case where the value process Y is multidimensional and the generator g can grow arbitrarily fast in the control process Z. Our focus is on Markovian systems, in which the functions b, σ, g and h are deterministic. We consider generators that are Lipschitz continuous in X and Y and locally Lipschitz continuous in Z. For one-dimensional value processes the decoupled system (with b depending only on X) has been solved by Cheridito and Nam [6] based on Malliavin calculus arguments. In fact, using that for Lipschitz continuous generators the trace of the Malliavin derivative of the value process Y is a version of the control process, they show that the control process can be uniformly bounded, hence enabling solvability for locally Lipschitz generators by a truncation argument. To solve (1.1) in the multidimensional case we propose a Picard iteration scheme which yields a Cauchy sequence in an appropriate Banach space under uniform boundedness of the control processes. Using Malliavin calculus arguments the boundedness is guaranteed if the time horizon is small enough. Here we make ample use of the method in Cheridito and Nam [6]. Moreover using the PDE representation of Markovian Lipschitz FBSDEs as developed for instance in Delarue [8] and a pasting procedure, we construct a unique global solution for generators with an additional monotonicity-type condition and non-degeneracy of the volatility σ, see Theorem 2.5.

Systems such as (1.1) naturally appear in numerous areas of applied mathematics including stochastic control and mathematical finance, see e.g. Yong and Zhou [30], El Karoui et al. [12], Horst et al. [17], Kramkov and Pulido [22] and Bielagk et al. [3]. As shown for instance in Ma et al. [25] and Cheridito and Nam [6], in the Markovian case, FBSDEs can be linked to parabolic PDEs. More recently it is shown in Fromm et al. [15] that FBSDEs can be used in the study of the Skorokhod embedding problem.
BSDEs and FBSDEs with Lipschitz continuous generators are well understood, we refer to El Karoui et al. [12] and Delarue [8]. If Y is one-dimensional and g has quadratic growth in the control process Z, BSDE solutions have been obtained by Kobylanski [21], Barrieu and El Karoui [2] and Briand and Hu [4, 5] under different assumptions on the terminal condition $\xi = h(X_T)$. We further refer to Delbaen et al. [9], Drapeau et al. [11], Cheridito and Nam [6] and Heyne et al. [16] for results on one-dimensional BSDEs and FBSDEs with superquadratic growth. Mainly due to the absence of comparison principle, general solvability of multidimensional BSDEs with quadratic growth is less well understood. Under smallness of the terminal condition solvability is shown in Tevzadze [28], see also Hu and Tang [18], Luo and Tangpi [24], Jamneshan et al. [20], Cheridito and Nam [7], Frei [13] and Xing and Žitković [29] for more recent developments.
To the best of our knowledge, Antonelli and Hamadène [1], Luo and Tangpi [24] and Fromm and Imkeller [14] are the only works studying well-posedness of coupled FBSDEs with quadratic growth. In Antonelli and Hamadène [1] the authors consider a one-dimensional equation with one dimensional Brownian motion and impose monotonicity conditions on the coefficient so that comparison principles for SDEs and BSDEs can be applied. A (non-necessarily unique) solution is then obtained by monotone convergence of an iterative scheme. This approach cannot be transfered to the present multidimensional case since comparison results are not available. Fromm and Imkeller [14] consider fully coupled Markovian FBSDEs with multidimensional forward and value processes and locally Lipschitz continuous generators in (Y, Z). Using the technique of decoupling fields they obtain existence of a unique local solution and provide an extension to a maximal time interval. Compared to Fromm and Imkeller [14], we use an essentially different technique based on Malliavin calculus which guarantees the existence of a uniformly Lipschitz decoupling field and Malliavin differentiability of solutions. Moreover, we also construct a global solution. Although the non-Markovian system studied in Luo and Tangpi [24] is the same as the one considered here, the techniques are essentially different. In particular, the growth conditions in the present paper are weaker and we do not impose any diagonally quadratic condition. Our main results can be extended to the non-Markovian setting and to random diffusion coefficient (when σ depends on X and Y) under stronger assumptions involving the Malliavin derivatives of g and h, for details we refer to the Ph.D. thesis of Luo [23].
The paper is organized as follows. In the next section, we present the setting and main results. In Section 3 we prove local solvability of multidimensional BSDE with superquadratic growth and give conditions guaranteeing global solvability. Section 4 is dedicated to the proofs of the main results.

2 Main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented filtration generated by a d-dimensional Brownian motion W, and $\mathcal{F} = \mathcal{F}_T$ for a finite time horizon $T \in (0, \infty)$. The product $\Omega \times [0, T]$ is endowed with the predictable σ-algebra. Subsets of $\mathbb{R}^k$ and $\mathbb{R}^{k\times k}$, $k \in \mathbb{N}$,

are always endowed with the Borel σ-algebra induced by the Euclidean norm $|\cdot|$. The interval $[0,T]$ is equipped with the Lebesgue measure. Unless otherwise stated, all equalities and inequalities between random variables and processes will be understood in the P-almost sure and $P\otimes dt$-almost sure sense, respectively. For $p\in[1,\infty]$ and $k\in\mathbb{N}$, we denote by $\mathcal{S}^p(\mathbb{R}^k)$ the space of all predictable continuous processes X with values in $\mathbb{R}^k$ such that $\|X\|_{\mathcal{S}^p(\mathbb{R}^k)} := ||\sup_{t\in[0,T]}|X_t|\,||_p<\infty$, and by $\mathcal{H}^p(\mathbb{R}^k)$ the space of all predictable processes Z with values in $\mathbb{R}^k$ such that $\|Z\|_{\mathcal{H}^p(\mathbb{R}^k)} := ||(\int_0^T |Z_u|^2\,du)^{1/2}||_p<\infty$. Here, $||\cdot||_p$ denotes the L^p-norm.
Let $l,m\in\mathbb{N}$ be fixed. A solution of (1.1) with values in $\mathbb{R}^m\times\mathbb{R}^l\times\mathbb{R}^{l\times d}$ can be obtained under the following conditions:

(A1) $b:[0,T]\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^m$ is a measurable function, $b_t(\cdot,\cdot)$ is continuous for each $t\in[0,T]$, and there exist $k_1,k_2,\lambda_1\geq 0$ such that

$$\left|b_t(x,y)-b_t(x',y')\right|\leq k_1\left|x-x'\right|+k_2\left|y-y'\right| \text{ and } |b_t(x,y)|\leq\lambda_1(1+|x|+|y|)$$

for all $x,x'\in\mathbb{R}^m$ and $y,y'\in\mathbb{R}^l$.

(A2) $\sigma:[0,T]\to\mathbb{R}^{m\times d}$ is a measurable function and there is $\lambda_2\geq 0$ such that $|\sigma_t|\leq\lambda_2$ for all $t\in[0,T]$.

(A3) $h:\mathbb{R}^m\to\mathbb{R}^l$ is a continuous function and there exists $k_5\geq 0$ such that

$$\left|h(x)-h(x')\right|\leq k_5\left|x-x'\right|$$

for all $x,x'\in\mathbb{R}^m$.

(A4) $g:[0,T]\times\mathbb{R}^m\times\mathbb{R}^l\times\mathbb{R}^{l\times d}\to\mathbb{R}^l$ is a measurable function and $\int_0^T|g_t(0,0,0)|\,dt<+\infty$. Moreover, the function $g_t(\cdot,\cdot,\cdot)$ is continuous for each $t\in[0,T]$ and there exist $k_3,k_4\geq 0$ and a nondecreasing function $\rho:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\left|g_t(x,y,z)-g_t(x',y,z)\right|\leq k_3\left|x-x'\right| \tag{2.1}$$

for all $x,x'\in\mathbb{R}^m$, $y\in\mathbb{R}^l$ and $z\in\mathbb{R}^{l\times d}$ such that $|z|\leq M:=8\lambda_2k_5\sqrt{dl}$ and

$$\left|g_t(x,y,z)-g_t(x,y',z')\right|\leq k_4\left|y-y'\right|+\rho\left(|z|\vee\left|z'\right|\right)\left|z-z'\right|$$

for all $x\in\mathbb{R}^m$, $y,y'\in\mathbb{R}^l$ and $z,z'\in\mathbb{R}^{l\times d}$.

(A5) There exists a constant $K\geq 0$ such that

$$\left|g_t(x,y,z)-g_t(x',y,z)-g_t(x,y',z')+g_t(x',y',z')\right|\leq K\left|x-x'\right|\left(\left|y-y'\right|+\left|z-z'\right|\right)$$

for all $t\in[0,T]$, $x,x'\in\mathbb{R}^m$, $y,y'\in\mathbb{R}^l$ and $z,z'\in\mathbb{R}^{l\times d}$.

Our main result ensures local existence and uniqueness for the coupled FBSDE (1.1) under the previous assumptions. The proof is given in Section 4.

Theorem 2.1. *Assume that (A1)-(A5) hold. Then there exists a constant $C > 0$ depending on k_1, k_2, k_3, k_4, k_5, λ_2, l and d, such that the FBSDE (1.1) has a unique solution $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ with $|Z_t| \le M$, whenever $T \le C$.*

Local existence results have been obtained in [14, Theorem 3] and [24, Theorem 2.1] in essentially different settings and with different methods. Let us mention that our technique allows to obtain existence of solutions of coupled FBSDEs with Burger type nonlinearities at least for small enough time horizons.

Example 2.2. Assume that T is small enough, b, σ and h satisfy (A1)-(A3), with $|h| \le \lambda_5$ for some $\lambda_5 \ge 0$. Then for each $k \ge 1$ the FBSDE

$$\begin{cases} X_t &= x + \int_0^t b_s(X_s, Y_s)\, ds + \int_0^t \sigma_s dW_s \\ Y_t &= h(X_T) + \int_t^T Y_s |Z_s|^k\, ds - \int_t^T Z_s dW_s, \quad t \in [0, T] \end{cases} \tag{2.2}$$

admits a solution $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$. The details are given in Subsection 4.2.$\diamondsuit$

Remark 2.3. The condition (A5) is the minimal condition needed to ensure Lipschitz continuity in y, z of the Malliavin derivative of $g_t(X_t, y, z)$ for a given SDE solution X, see e.g. El Karoui et al. [12] and Cheridito and Nam [6] for details. When the generator g is of the form $g_t(x, y, z) := f_t^1(x) + f_t^2(y) + f_t^3(z)$ for some functions f^1, f^2 and f^3, then (A5) is satisfied.
Moreover, let us mention that an advantage of our method is that it implies Malliavin differentiability of the forward process X and the value process Y in the solution (X, Y, Z) of the FBSDE (1.1) obtained in Theorem 2.1. We refer to Subsection 4.3 for details. ♦

Remark 2.4. The following counterexample shows that in general, even in the one-dimensional case, coupled systems do not have a unique global solution. Consider the FBSDE

$$\begin{cases} X_t = \int_0^t Y_u\, du \\ Y_t = \int_t^T kX_u\, du - \int_t^T Z_u\, dW_u. \end{cases}$$

This equation can be rewritten as

$$Y_t = \int_t^T \int_0^s kY_u\, du\, ds - \int_t^T Z_u\, dW_u. \tag{2.3}$$

It is shown in [10, Example 3.2] that if $T\sqrt{k} < \frac{\pi}{2}$ then the BSDE with time-delayed generator (2.3) has a unique solution whereas if $T\sqrt{k} = \frac{\pi}{2}$, the equation (2.3) may not have any solution and if it has one, there are infinitely many. ♦

Next, we would like to find conditions under which Theorem 2.1 can be extended to obtain global solvability. In the present setting, under additional assumptions, a pasting method based on PDEs allows to get global existence and uniqueness for the FBSDE (1.1).

(A6) There exist $K_4 \geq 0$ satisfying $K_4 \geq 2e^{2k_1T}k_2^2T(k_5^2 + k_3T) + k_3 + \rho^2(\bar{M}\sqrt{dl})$ with $\bar{M} = 8\lambda_2 K_5\sqrt{dl}$ and $K_5 = \sqrt{2(k_5^2 + k_3T)}e^{k_1T}$ such that

$$\sum_{i=1}^{l}(y^i - y'^i)\left(g_t^i(x,y,z) - g_t^i(x,y',z)\right) \leq -K_4\left|y - y'\right|^2$$

for all $x \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^l$ and $z \in \mathbb{R}^{l\times d}$.

(A7) There exist $K_1, K_4 \geq 0$ satisfying $\sqrt{2K_1k_5^2(K_4 - \rho^2(M\sqrt{dl}))} \geq k_2k_5^2 + k_3$ such that

$$\sum_{i=1}^{m}(x^i - x'^i)\left(b_t^i(x,y) - b_t^i(x',y)\right) \leq -K_1\left|x - x'\right|^2,$$

$$\sum_{i=1}^{l}(y^i - y'^i)\left(g_t^i(x,y,z) - g_t^i(x,y',z)\right) \leq -K_4\left|y - y'\right|^2$$

for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^l$ and $z \in \mathbb{R}^{l\times d}$.

Theorem 2.5. *Assume that (A1)-(A5) hold and there exist $\lambda_3, \lambda_4, \lambda_5 > 0$ such that*[1]

$$\begin{cases} |b_t(x,y)| & \leq \lambda_1(1 + |y|) \\ \langle x, \sigma_t\sigma_t^* x\rangle & \geq \lambda_3|x|^2 \\ |g_t(x,y,z)| & \leq \lambda_4(1 + |y| + \rho(|z|)\,|z|) \\ |h(x)| & \leq \lambda_5 \end{cases} \tag{2.4}$$

for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^l$ and $z, z' \in \mathbb{R}^{l\times d}$. Then, if (A6) (respectively (A7)) is satisfied, the FBSDE (1.1) has a unique global solution $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ such that $|Z_t| \leq \bar{M}$ (respectively $|Z_t| \leq M$).

Remark 2.6. Assumptions (A6) and (A7) can be understood as monotonicity conditions on the generator and the drift coefficient. These conditions are satisfied for instance when the components of g (resp. b) are linear in y (resp. x). In fact, if there is a function f with values in $\mathbb{R}^l$ such that $g_t(x,y,z) := -K_4y + f_t(x,z)$, then g satisfies (A6), and suitable conditions on f guarantee that g satisfies (A4) and (A5) as well. ♦

Notice that Theorem 2.5 yields existence of a decoupling field, see [14]. In particular, the boundedness of Z yields uniform Lipschitz continuity of the decoupling field.
Theorem 2.1 relies on an existence result for multidimensional BSDEs presented in Nam [26] and revisited in the next section.

[1] σ_t^* is the transpose matrix of σ_t.

3 Multidimensional BSDEs with bounded Malliavin derivatives

Let us introduce the spaces of Malliavin differentiable random variables and stochastic processes $\mathcal{D}^{1,2}(\mathbb{R}^l)$ and $\mathcal{L}_a^{1,2}(\mathbb{R}^l)$. For a thorough treatment of the theory of Malliavin calculus we refer to Nualart [27]. Let $\mathcal{M}$ be the class of smooth random variables $\xi = (\xi^1, \dots, \xi^l)$ of the form

$$\xi^i = \varphi^i\Big(\int_0^T h_s^{i1}\, dW_s, \dots, \int_0^T h_s^{in}\, dW_s\Big)$$

where φ^i is in the space $C_p^\infty(\mathbb{R}^n;\mathbb{R})$ of infinitely continuously differentiable functions whose partial derivatives have polynomial growth, $h^{i1}, \dots, h^{in} \in L^2([0,T];\mathbb{R}^d)$ and $n \geq 1$. For every ξ in $\mathcal{M}$ let the operator $D = (D^1, \dots, D^d) : \mathcal{M} \to L^2(\Omega \times [0,T];\mathbb{R}^d)$ be given by

$$D_t\xi^i := \sum_{j=1}^n \frac{\partial \varphi^i}{\partial x_j}\Big(\int_0^T h_s^{i1}\, dW_s, \dots, \int_0^T h_s^{in}\, dW_s\Big) h_t^{ij}, \quad 0 \leq t \leq T,\ 1 \leq i \leq l,$$

and the norm $\|\xi\|_{1,2} := (E[|\xi|^2 + \int_0^T |D_t\xi|^2\, dt])^{1/2}$. As shown in Nualart [27], the operator D extends to the closure $\mathcal{D}^{1,2}(\mathbb{R}^l)$ of the set $\mathcal{M}$ with respect to the norm $\|\cdot\|_{1,2}$. A random variable ξ is Malliavin differentiable if $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ and we denote by $D_t\xi$ its Malliavin derivative. Denote by $\mathcal{L}_a^{1,2}(\mathbb{R}^l)$ the space of processes $Y \in \mathcal{H}^2(\mathbb{R}^l)$ such that $Y_t \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ for all $t \in [0,T]$, the process DY_t admits a square integrable progressively measurable version and

$$\|Y\|^2_{\mathcal{L}_a^{1,2}(\mathbb{R}^l)} := \|Y\|_{\mathcal{H}^2(\mathbb{R}^l)} + E\Big[\int_0^T\int_0^T |D_rY_t|^2\, dr\, dt\Big] < \infty.$$

We next consider a system of superquadratic BSDEs of the form

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u)du - \int_t^T Z_u dW_u \tag{3.1}$$

satisfying the following conditions:

(B1) $g : \Omega \times [0,T] \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \to \mathbb{R}^l$ is a measurable function and there exist a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|g_t(y,z) - g_t(y',z')\right| \leq B\left|y - y'\right| + \rho\left(|z| \vee \left|z'\right|\right)\left|z - z'\right|$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}^l$ and $z, z' \in \mathbb{R}^{l\times d}$.

(B2) $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ and there exist constants $A_{ij} \geq 0$ such that $|D_t^j\xi^i| \leq A_{ij}$ for all $i = 1, \dots, l$, $j = 1, \dots, d$ and $t \in [0,T]$.

(B3) $g_\cdot(0,0) \in \mathcal{H}^4(\mathbb{R}^l)$ and there exist Borel-measurable functions $q_{ij} : [0,T] \to \mathbb{R}_+$ satisfying $\int_0^T q_{ij}^2(t)dt < \infty$ such that for every pair $(y,z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ with

$$|z| \leq Q := \sqrt{2\sum_{j=1}^{d}\left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l}\int_0^T q_{ij}^2(t)dt\right)}$$

one has

- $g_\cdot(y,z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^l)$ with $|D_u^j g_t^i(y,z)| \leq q_{ij}(t)$ for all $i = 1,\dots,l$, $j = 1,\dots,d$ and $u \in [0,T]$,
- for almost all $u \in [0,T]$ one has

$$\left|D_u g_t(y,z) - D_u g_t(y',z')\right| \leq K_u(t)\left(\left|y-y'\right| + \left|z-z'\right|\right)$$

  for all $t \in [0,T]$, $y,y' \in \mathbb{R}^l$ and $z,z' \in \mathbb{R}^{l\times d}$ for some $\mathbb{R}_+$-valued adapted process $(K_u(t))_{t\in[0,T]}$ satisfying $\int_0^T \|K_u\|_{\mathcal{H}^4(\mathbb{R})}^4\, du < \infty$.

The following is an extension of Cheridito and Nam [6, Theorem 2.2] to the multidimensional case. It was proved in Nam [26] under slightly different assumptions. For instance, we do not assume a monotonicity-type condition on $y \mapsto g_t(y,z)$ for every z. Our results rely on the techniques of [6]. For the sake of completeness we give the proof.

Theorem 3.1. *Assume that (B1)-(B3) hold and $T \leq \frac{\log(2)}{2B+\rho^2(Q)+1}$. Then the BSDE (3.1) admits a unique solution in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ and $|Z_t| \leq Q$.*

Consider the following stronger versions of the conditions (B1) and (B3):

(B1') g is continuously differentiable in (y,z) and there exist constants $B \in \mathbb{R}_+$ and $\rho \in \mathbb{R}_+$ such that $|\partial_y g_t(y,z)| \leq B$ and $|\partial_z g_t(y,z)| \leq \rho$ for all $t \in [0,T]$, $y,y' \in \mathbb{R}^l$ and $z,z' \in \mathbb{R}^{l\times d}$.

(B3') The condition (B3) holds for all $(y,z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$.

Lemma 3.2. *If (B1'), (B2) and (B3') hold, then the BSDE (3.1) admits a unique solution $(Y,Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l\times d})$ and*

$$|Z_t^j|^2 \leq \left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l}\int_t^T q_{ij}^2(s)e^{-(2B+\rho^2+1)(T-s)}ds\right)e^{(2B+\rho^2+1)(T-t)} \quad \textit{for all } j = 1,\dots,d. \quad (3.2)$$

Proof. By (B2), each component ξ^i of ξ has bounded Malliavin derivative, which implies by [6, Lemma 2.5] that $E[|\xi^i|^p] < \infty$ for all $p \geq 1$. It follows from El Karoui et al. [12, Theorem 5.1 and Proposition 5.3] that the BSDE (3.1) has a unique solution $(Y,Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l\times d})$, which is Malliavin differentiable. Moreover for every $i = 1,\dots,l$ and $j = 1,\dots,d$, the process $(D_r^j Y_t^i, D_r^j Z_t^i)_{t\in[0,T]}$ has a version $(U_t^{ij,r}, V_t^{ij,r})_{t\in[0,T]}$ which satisfies

$$U_t^{ij,r} = 0, \quad V_t^{ij,r} = 0, \quad \text{for} \quad 0 \leq t < r \leq T,$$

and is the unique solution in $\mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l\times d})$ of the BSDE

$$U_t^{j,r} = D_r^j \xi + \int_t^T \partial_y g_s(Y_s, Z_s) U_s^{j,r} + \partial_z g_s(Y_s, Z_s) V_s^{j,r} + D_r^j g_s(Y_s, Z_s) ds - \int_t^T V_s^{j,r} dW_s.$$

Applying Itô's formula to $|U_t^{j,r}|^2$ yields

$$\begin{aligned}
|U_t^{j,r}|^2 &= |D_r^j \xi|^2 - \int_t^T 2U_s^{j,r} V_s^{j,r} dW_s \\
&\quad + \int_t^T 2U_s^{j,r} \partial_y g_s(Y_s, Z_s) U_s^{j,r} + 2U_s^{j,r} \partial_z g_s(Y_s, Z_s) V_s^{j,r} + 2U_s^{j,r} D_r^j g_s(Y_s, Z_s) - |V_s^{j,r}|^2 ds \\
&\leq |D_r^j \xi|^2 - \int_t^T 2U_s^{j,r} V_s^{j,r} dW_s + \int_t^T 2B|U_s^{j,r}|^2 + 2\rho |U_s^{j,r}||V_s^{j,r}| + 2\sqrt{\sum_{i=1}^l q_{ij}^2(s)} |U_s^{j,r}| - |V_s^{j,r}|^2 ds \\
&\leq |D_r^j \xi|^2 - \int_t^T 2U_s^{j,r} V_s^{j,r} dW_s + \int_t^T \left(2B + \rho^2 + 1\right) |U_s^{j,r}|^2 + \sum_{i=1}^l q_{ij}^2(s) ds.
\end{aligned}$$

Using condition (B3) and taking conditional expectation in the above inequality yields

$$|U_t^{j,r}|^2 \leq E\Big[\sum_{i=1}^l A_{ij}^2 + \int_t^T \left(2B + \rho^2 + 1\right) |U_s^{j,r}|^2 + \sum_{i=1}^l q_{ij}^2(s) ds \,\Big|\, \mathcal{F}_t\Big]. \tag{3.3}$$

By El Karoui et al. [12, Proposition 5.3] the process Z is a version of the trace $(U_t^t)_{t\in[0,T]}$ of the Malliavin derivative of Y. Hence (3.2) follows from (3.3) by applying Gronwall's inequality. □

Proof (Theorem 3.1). Define the Lipschitz continuous function $\tilde{g}$ by

$$\tilde{g}_t(y, z) = \begin{cases} g_t(y, z) & \text{if} \quad |z| \leq Q, \\ g_t(y, Qz/|z|) & \text{if} \quad |z| > Q. \end{cases} \tag{3.4}$$

By Cheridito and Nam [6, Lemma 2.5], the condition (B2) implies $E\left[|\xi|^p\right] < +\infty$ for all $p \in [1, \infty)$. Thus, $\xi \in L^p$ for all $p \geq 1$. Therefore, it follows from El Karoui et al. [12, Theorem 5.1] that the BSDE corresponding to $(\tilde{g}, \xi)$ has a unique solution $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l\times d})$. For $x = (y, z) \in \mathbb{R}^{l+l\times d}$ let $\beta \in C^\infty(\mathbb{R}^{l+l\times d})$ be the mollifier

$$\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant $\lambda \in \mathbb{R}_+$ is chosen such that $\int_{\mathbb{R}^{l+l\times d}} \beta(x)dx = 1$. Set $\beta^n(x) := n^{l+l\times d}\beta(nx)$, $n \in \mathbb{N} \setminus \{0\}$, and define

$$g_t^n(\omega, x) := \int\limits_{\mathbb{R}^{l+l\times d}} \tilde{g}_t(\omega, x')\beta^n(x-x')dx'$$

so that for each $n > 0$ the function g^n satisfies (B1') and (B3') with the constant ρ replaced by $\rho(Q)$. By Lemma 3.2 the BSDE corresponding to (g^n, ξ) has a unique solution (Y^n, Z^n) in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l\times d})$ which satisfies

$$\begin{aligned}
|Z_t^{n,j}|^2 &\le \left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l} \int\limits_t^T q_{ij}^2(s) e^{-\left(2B+\rho^2(Q)+1\right)(T-s)} ds \right) e^{\left(2B+\rho^2(Q)+1\right)(T-t)} \\
&\le \left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l} \int\limits_0^T q_{ij}^2(s) ds \right) e^{\left(2B+\rho^2(Q)+1\right)T}.
\end{aligned}$$

Since $T \le \frac{\log(2)}{2B+\rho^2(Q)+1}$ we obtain

$$|Z_t^{n,j}|^2 \le 2 \left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l} \int\limits_0^T q_{ij}^2(s) ds \right) \quad \text{for all } j = 1, \dots, d.$$

This shows $|Z_t^n| \le Q$. Since g^n converges uniformly in (t, ω, y, z) to $\tilde{g}$, using the procedure of the proof of Cheridito and Nam [6, Theorem 2.2], it follows that (Y^n, Z^n) converges to (Y, Z) in $\mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l\times d})$, so that $|Z_t| \le Q$. Since $\tilde{g}(y, z) = g(y, z)$ for all $(y, z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ with $|z| \le Q$, it follows that (Y, Z) is the unique solution of the BSDE corresponding to (ξ, g) in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$. □

Corollary 3.3. *Suppose (B1)-(B3) hold, $T \le \frac{\log(2)}{2B+\rho^2(Q)+1}$ and $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ is the solution of the BSDE (3.1). Then $Y_t \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ for all $t \in [0, T]$ and for every $j = 1, \dots, d$, one has*

$$|D_r^j Y_t|^2 \le 2 \left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l} \int\limits_0^T q_{ij}^2(s) ds \right) \quad \text{for all } r \in [0, t]. \tag{3.5}$$

Proof. Since $|Z| \le Q$ is bounded, (Y, Z) solves the BSDE with terminal condition ξ and generator $\tilde{g}$ defined by (3.4). If g satisfies (B1') and (B3'), then the result follows from Lemma 3.2. Otherwise consider the sequence of smooth functions g^n converging to g as defined in the proof of Theorem 3.1. Let $(Y^n, Z^n) \in \mathcal{S}^4(\mathbb{R}^l) \times \mathcal{H}^4(\mathbb{R}^{l\times d})$ be the solutions to the BSDEs corresponding to (g^n, ξ), which converge to (Y, Z) in $\mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l\times d})$. By Lemma 3.2 $(Y_t^n, Z_t^n) \in \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l\times d})$ for each $t \in [0, T]$ and the arguments in the proof of Theorem 3.1 imply

$$|D_r^j Y_t^n|^2 \le 2 \left(\sum_{i=1}^{l} A_{ij}^2 + \sum_{i=1}^{l} \int\limits_0^T q_{ij}^2(s) ds \right) \quad j = 1, \dots, d, \;\; r, t \in [0, T].$$

Hence, $\sup_{n\in\mathbb{N}} E[\int_0^T |D_r^j Y_t^n|^2\,dr] < \infty$ for each $t \in [0,T]$. Since (Y_t^n) converges to Y_t in L^2, it follows from Nualart [27, Lemma 1.2.3] that $Y_t \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ and $(D_r Y_t^n)$ converges to $D_r Y_t$ in the weak topology of $\mathcal{H}^2(\mathbb{R}^{l\times d})$. Thus, $D_r Y_t$ satisfies (3.5). □

As a consequence to Theorem 3.1, we give a condition for global solvability of fully coupled systems of BSDEs. For the remainder of this section we put

$$\Delta_n := \frac{\log(2)}{2B + \rho^2(2^n Q) + 1}, \quad n \in \mathbb{N}.$$

Proposition 3.4. *Assume that (B1)-(B2) hold, that there exists $N \in \mathbb{N}$ such that $\sum_{n=0}^N \Delta_n \geq T$, and (B3) holds with Q replaced by $2^N Q$. Then the BSDE (3.1) has a unique solution in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ and $|Z_t| \leq 2^N Q$.*

Proof. If $T \leq \Delta_0$ then the result follows from Theorem 3.1. Otherwise, if $T > \Delta_0$ it follows by the same arguments as in the proof of Theorem 3.1 that the BSDE (3.1) has a unique solution (Y^0, Z^0) in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ on the interval $[T - \Delta_0, T]$. Moreover, Z^0 satisfies $|Z_t^0| \leq Q$ and by Corollary 3.3 one has $Y^0_{T-\Delta_0} \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ and for every $r \leq T - \Delta_0$,

$$|D_r^j Y^0_{T-\Delta_0}|^2 \leq \sum_{i=1}^l 2|A_{ij}|^2 + \sum_{i=1}^l \int_0^T 2|q_{ij}(t)|^2 dt \quad \text{for all } j = 1, \dots, d.$$

Since g satisfies (B3) for all $(y,z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ such that $|z| \leq cQ$, again by Theorem 3.1 the BSDE (3.1) with terminal condition $Y^0_{T-\Delta_0}$ has a unique solution (Y^1, Z^1) in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ on $[(T - \Delta_0 - \Delta_1) \vee 0, T - \Delta_0]$, and

$$\begin{aligned}
&|D_r^j Y^1_{(T-\Delta_0-\Delta_1)\vee 0}|^2 \leq \sum_{i=1}^l 2^2|A_{ij}|^2 + \sum_{i=1}^l \int_0^T (2^2+2)|q_{ij}(t)|^2 dt, \quad \text{for all } j = 1, \dots, d\\
&|Z_t^1| \leq 2Q, \quad t \in [(T - \Delta_0 - \Delta_1) \vee 0, T - \Delta_0]\,.
\end{aligned}$$

Repeating the previous arguments, for $N \geq 2$ the BSDE (3.1) has a unique solution (Y^N, Z^N) in $\mathcal{S}^4(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ on $[(T - \sum_{n=0}^N \Delta_n) \vee 0, (T - \sum_{n=0}^{N-1} \Delta_n) \vee 0]$ with terminal condition $Y_{(T-\sum_{n=0}^{N-1}\Delta_n)\vee 0}$. Moreover,

$$\begin{aligned}
&|D_r^j Y^N_{(T-\sum_{n=0}^N \Delta_n)\vee 0}|^2 \leq \sum_{i=1}^l 2^N|A_{ij}|^2 + \sum_{i=1}^l \int_0^T (\sum_{k=1}^N 2^k)|q_{ij}(t)|^2 dt \quad \text{for all } j = 1, \dots, d\\
&|Z_t^N| \leq 2^N Q, \quad t \in \Big[(T - \sum_{n=0}^N \Delta_n) \vee 0, (T - \sum_{n=0}^{N-1} \Delta_n) \vee 0\Big].
\end{aligned}$$

Hence, the pair (Y, Z) given by

$$Y := Y^0 1_{[T-\Delta_1,T]} + \sum_{n=1}^{N} Y^n 1_{[(T-\sum_{i=0}^{n}\Delta_i)\vee 0,(T-\sum_{i=0}^{n-1}\Delta_i)\vee 0]}$$
$$Z := Z^0 1_{[T-\Delta_1,T]} + \sum_{n=1}^{N} Z^n 1_{[(T-\sum_{i=0}^{n}\Delta_i)\vee 0,(T-\sum_{i=0}^{n-1}\Delta_i)\vee 0]}$$

solves (3.1) and its uniqueness follows from Theorem 3.1. □

Remark 3.5. The condition $\sum_{n=0}^{N}\Delta_n \geq T$ for some $N \in \mathbb{N}$ does not guarantee global solvability of multidimentional BSDEs with superquadratic growth. In fact, if $\rho(x) \geq C(1+\sqrt{x})$ for all $x \geq 0$, then $\sum_{n\geq 0}\Delta_n < \infty$. However, it does guarantee global solvability for BSDEs whose generator grows slightly faster than the linear function. For instance, if $\rho(x) \leq C(1+\sqrt{\log(1+x)})$ one has

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\log(2)}{2B+\rho^2(2^N Q)+1} &\geq \sum_{n=0}^{\infty}\frac{\log(2)}{2B+2C^2(1+\log(2^N(1+Q)))+1}\\ &= \sum_{n=0}^{\infty}\frac{\log(2)}{2B+2C^2(1+\log(1+Q)+n\log(2))+1} = \infty.\end{aligned}$$

Note also that global solvability of strict subquadratic systems has been established by Cheridito and Nam [7]. ♦

4 Coupled FBSDE with superquadratic growth

4.1 Proof of Theorem 2.1

Step 1: We first assume that h, b and g are continuously differentiable in all variables. Let us define

$$C_1 := \frac{k_5^2}{k_3^2} \wedge \frac{\log 2}{k_1} \wedge \frac{\lambda_2}{k_2 M} \wedge \frac{\log 2}{2k_4+\rho^2(M)+1}$$

with $M := 8k_5\lambda_2\sqrt{dl}$. We will show that for $T \leq C_1$, the sequence (X^n, Y^n, Z^n) given by $X^0 = 0$, $Y^0 = 0$, $Z^0 = 0$ and

$$\begin{cases} X_t^{n+1} &= x + \int_0^t b(X_u^{n+1}, Y_u^n)\,du + \int_0^t \sigma_u\,dW_u \\ Y_t^{n+1} &= h(X_T^{n+1}) + \int_t^T g_u(X_u^{n+1}, Y_u^{n+1}, Z_u^{n+1})\,du - \int_t^T Z_u^{n+1}\,dW_u, \quad n \geq 1 \end{cases}$$

is well defined and that $|Z_t^n| \leq M$ for all $n \in \mathbb{N}$ and $t \in [0,T]$. The process X^1 is well defined, X_t^1 belongs to $\mathcal{D}^{1,2}(\mathbb{R}^m)$ for every t and the process $(D_r X_t)_{t\in[0,T]}$ satisfies the linear equation

$$D_r X_t^1 = 0,\ 0 \leq t < r \leq T,$$
$$D_r X_t^1 = \int_r^t (\partial_x b D_r X_u^1 + \partial_y b D_r Y_u^0)\,du + D_r\left(\int_r^t \sigma_u\,dW_u\right),\ 0 \leq r \leq t \leq T,$$

with $D_r(\int_r^t \sigma_u\, dW_u) = \sigma 1_{[r,t]}$, see Nualart [27, Lemma 2.2.1 and Theorem 2.2.1]. Hence, since b is Lipschitz continuous, we have

$$\left|D_r X_t^1\right| \leq \int_r^t k_1 D_r X_u^1\, du + \sigma_r \quad \text{and} \quad \left|D_r X_t^1\right| \leq \lambda_2 e^{Tk_1},$$

where the second estimate comes from Gronwall's inequality. We will now show that since $T \leq C_1$, $h(X_T^1)$ and $g(X^1,\cdot,\cdot)$ satisfy (B1)-(B3). In fact, since h is continuously differentiable and $X_T^1 \in \mathcal{D}^{1,2}(\mathbb{R}^m)$, it follows from the chain rule, see for instance Nualart [27, Proposition 1.2.4], that $h(X_T^1) \in \mathcal{D}^{1,2}(\mathbb{R}^l)$ and $|D_r^j(h(X_T^1))| = |\partial_x h(X_T^1) D_r^j X_T^1| \leq \lambda_2 k_5 e^{Tk_1}$ for all $r \in [0,T]$, $j = 1,\dots,d$. Using $T \leq \frac{\log 2}{k_1}$, we deduce that $h(X_T^1)$ satisfies (B2) with $A_{ij} := 2\lambda_2 k_5$. Similarly, by (A4) and using that the function $x \mapsto g(x,y,z)$ is continuously differentiable, it follows that $g_\cdot(X_\cdot^1,y,z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^l)$ and $\left|D^j(g_\cdot^i(X^1,y,z))\right| \leq \lambda_2 k_3 e^{Tk_1}$, $j = 1,\dots,d$ for all $(y,z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ such that $|z| \leq M$ and, due to (A5), applying the same argument to $\hat{g}_t(x,y,y',z,z') := g_t(x,y,z) - g_t(x,y',z')$ yields

$$|D_r^j g_t(X_t^1,y,z) - D_r^j g_t(X_t^1,y',z')| \leq K\lambda_2 e^{Tk_1}.$$

Using $T \leq \frac{k_5^2}{k_3^2} \wedge \frac{\log 2}{k_1}$, we deduce that $g_\cdot(X^1,y,z)$ satisfies (B3) with $q_{ij} = 2\lambda_2 k_3$ and $K_u(t) := 2K\lambda_2$. Moreover due to (A4), the function $(t,y,z) \mapsto g_t(X_t^1,y,z)$ satisfies (B1).
Therefore, by $T \leq \frac{\log 2}{2k_4+\rho^2(M)+1}$, Theorem 3.1 ensures that (Y^1,Z^1) exists. Consider the function $\tilde{g}$ defined by

$$\tilde{g}_t(x,y,z) = \begin{cases} g_t(x,y,z) & \text{if } |z| \leq M \\ g_t(x,y,zM/|z|) & \text{if } |z| > M. \end{cases}$$

Since (Y^1,Z^1) also solves the BSDE with terminal condition $h(X_T^1)$ and a Lipschitz generator $\tilde{g}(X^1,\cdot,\cdot)$, it follows from Lemma 3.2 and its proof that $(Y_t^1,Z_t^1) \in \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l\times d})$ for all $t \in [0,T]$ and $D_t Y^1$ is bounded and it holds $Z_t^1 = D_t Y_t^1$. In addition, we have $\left|D_r X_t^1\right| \leq 4\lambda_2$ and $\left|D_r Y_t^1\right| \leq M$.
Now let $n \in \mathbb{N}$, assume that $(X_t^n,Y_t^n,Z_t^n) \in \mathcal{D}^{1,2}(\mathbb{R}^m) \times \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l\times d})$, $Z_t^n = D_t Y_t^n$ and $|D_r X_t^n| \leq 4\lambda_2$, $|D_r Y_t^n| \leq M$ for all $r,t \in [0,T]$. The process X^{n+1} is well defined, for each t; X_t^{n+1} belongs to $\mathcal{D}^{1,2}(\mathbb{R}^m)$ and it holds

$$\begin{aligned} &D_r X_t^{n+1} = 0,\ 0 \leq t < r \leq T, \\ &D_r X_t^{n+1} = \sigma_r + \int_r^t (\partial_x b D_r X_u^{n+1} + \partial_y b D_r Y_u^n)\, du,\ 0 \leq r \leq t \leq T. \end{aligned}$$

Since $\partial_x b$, $\partial_y b$ and σ are bounded by k_1, k_2 and λ_2 respectively, it follows from Gronwall's inequality that

$$\left|D_r X_t^{n+1}\right| \leq e^{Tk_1}\left(\lambda_2 + k_2 \int_0^T |D_r Y_u^n|\, du\right).$$

Hence,

$$\left|D_r X_t^{n+1}\right| \leq e^{Tk_1}(\lambda_2 + k_2 TM) < \infty \tag{4.1}$$

so that since $T \leq \frac{\lambda_2}{k_2 M}$, we have $\left|D_r X_t^{n+1}\right| \leq 4\lambda_2$. As above, $h(X_T^{n+1})$ and $g_\cdot(X^{n+1}, y, z)$ are Malliavin differentiable and satisfy (B1)-(B3) with $A_{ij} := 4\lambda_2 k_5$, $q_{ij} = 2\lambda_2 k_3$ and $K_u(t) := 2K\lambda_2$. It then follows again from Theorem 3.1 that (Y^{n+1}, Z^{n+1}) exists and $|Z^{n+1}| \leq M$ is bounded. Since (Y^{n+1}, Z^{n+1}) also solves the BSDE with terminal condition $h(X_T^{n+1})$ and Lipschitz continuous generator $\tilde{g}_t(X^{n+1}, \cdot, \cdot)$, Lemma 3.2 and its proof guarantee that $(Y_t^{n+1}, Z_t^{n+1}) \in \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l\times d})$ for all $t \in [0,T]$ and $D_t Y^{n+1}$ is bounded and it holds $Z_t^{n+1} = D_t Y_t^{n+1}$, with $\left|D_r Y_t^{n+1}\right| \leq M$.

Step 2: Now we show that there is a positive constant C_2 depending on k_1, k_2, k_3, k_4, k_5, λ_2, l and d, such that if $T \leq C_2$, then (X^n, Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l\times d})$. Using (A1) we can estimate the norm of the difference $X_t^{n+1} - X_t^n$ as

$$|X_t^{n+1} - X_t^n|^2 \leq 2\left(\int_0^t k_1|X_s^{n+1} - X_s^n|ds\right)^2 + 2\left(\int_0^t k_2|Y_s^n - Y_s^{n-1}|ds\right)^2.$$

Thus

$$\sup_{0\leq t\leq T}|X_t^{n+1} - X_t^n|^2 \leq 2\left(\int_0^T k_1|X_s^{n+1} - X_s^n|ds\right)^2 + 2\left(\int_0^T k_2|Y_s^n - Y_s^{n-1}|ds\right)^2.$$

Taking expectation on both sides and using Cauchy-Schwarz' inequality, we have

$$\begin{aligned} E\left[\sup_{0\leq t\leq T}|X_t^{n+1} - X_t^n|^2\right] &\leq 2Tk_1^2 E\left[\int_0^T |X_s^{n+1} - X_s^n|^2 ds\right] + 2Tk_2^2 E\left[\int_0^T |Y_s^n - Y_s^{n-1}|^2 ds\right] \\ &\leq 2T^2k_1^2 E\left[\sup_{0\leq t\leq T}|X_t^{n+1} - X_t^n|^2\right] + 2T^2k_2^2 E\left[\sup_{0\leq t\leq T}|Y_t^n - Y_t^{n-1}|^2\right]. \end{aligned}$$

Choosing T to be small enough so that $2T^2k_1^2 \leq \frac{1}{2}$, it follows

$$E\left[\sup_{0\leq t\leq T}|X_t^{n+1} - X_t^n|^2\right] \leq 4T^2k_2^2 E\left[\sup_{0\leq t\leq T}|Y_t^n - Y_t^{n-1}|^2\right]. \tag{4.2}$$

On the other hand, applying Itô's formula to $e^{\beta t}|Y_t^{n+1} - Y_t^n|^2$, $\beta \geq 0$, we have

$$\begin{aligned} e^{\beta t}|Y_t^{n+1} - Y_t^n|^2 = e^{\beta T}|h(X_T^{n+1}) - h(X_T^n)|^2 - 2\int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n)dW_s \\ - \int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds - \int_t^T \beta e^{\beta s}(Y_s^{n+1} - Y_s^n)^2 ds \\ + 2\int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)\left[g_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}) - g_s(X_s^n, Y_s^n, Z_s^n)\right] ds. \end{aligned}$$

Hence, due to the condition (A3) and the boundedness of (Z^n), it holds

$$\begin{aligned}
&e^{\beta t}|Y_t^{n+1}-Y_t^n|^2+\int_t^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2ds \\
&\leq e^{\beta T}\left|h(X_T^{n+1})-h(X_T^n)\right|^2-2\int_t^T e^{\beta s}(Y_s^{n+1}-Y_s^n)(Z_s^{n+1}-Z_s^n)dW_s \\
&\quad-\int_t^T \beta e^{\beta s}(Y_s^{n+1}-Y_s^n)^2ds+2\int_t^T e^{\beta s}\rho(M)\left|Y_s^{n+1}-Y_s^n\right|\left|Z_s^{n+1}-Z_s^n\right|ds \\
&\quad+2\int_t^T e^{\beta s}k_7\left|Y_s^{n+1}-Y_s^n\right|\left|X_s^{n+1}-X_s^n\right|ds+2\int_t^T e^{\beta s}k_4\left|Y_s^{n+1}-Y_s^n\right|^2ds.
\end{aligned}$$

With some positive constants α_1, α_2, it follows from (A3) and Young's inequality that

$$\begin{aligned}
&e^{\beta t}|Y_t^{n+1}-Y_t^n|^2+\int_t^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2ds\leq e^{\beta T}k_5^2|X_T^{n+1}-X_T^n|^2 \\
&-2\int_t^T e^{\beta s}(Y_s^{n+1}-Y_s^n)(Z_s^{n+1}-Z_s^n)dW_s+\alpha_2\int_t^T e^{\beta s}|X_s^{n+1}-X_s^n|^2ds \\
&+\left(\frac{(\rho(M))^2}{\alpha_1}+\frac{k_3^2}{\alpha_2}+2k_4-\beta\right)\int_t^T e^{\beta s}(Y_s^{n+1}-Y_s^n)^2ds+\alpha_1\int_t^T e^{\beta s}|Z_s^{n+1}-Z_s^n|^2ds. \qquad (4.3)
\end{aligned}$$

Letting $\beta=\frac{(\rho(M))^2}{\alpha_1}+\frac{k_3^2}{\alpha_2}+2k_4$ and taking expectation on both sides above, we have

$$\begin{aligned}
E\left[e^{\beta t}|Y_t^{n+1}-Y_t^n|^2\right]+E\left[\int_t^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2ds\right]&\leq e^{\beta T}k_5^2E\left[|X_T^{n+1}-X_T^n|^2\right] \\
&+\alpha_1E\left[\int_t^T e^{\beta s}|Z_s^{n+1}-Z_s^n|^2ds\right]+\alpha_2E\left[\int_t^T e^{\beta s}|X_s^{n+1}-X_s^n|^2ds\right].
\end{aligned}$$

Putting $\alpha_1=\frac{1}{2}$ and $\alpha_2=1$, the previous estimate yields

$$E\left[\int_0^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2ds\right]\leq 2e^{\beta T}k_5^2E\left[|X_T^{n+1}-X_T^n|^2\right]+2E\left[\int_0^T e^{\beta s}|X_s^{n+1}-X_s^n|^2ds\right].$$

Next, taking conditional expectation with respect to $\mathcal{F}_t$ in (4.3) gives

$$e^{\beta t}|Y_t^{n+1}-Y_t^n|^2+E\left[\int_t^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2ds\middle|\mathcal{F}_t\right]\leq e^{\beta T}k_5^2E\left[|X_T^{n+1}-X_T^n|^2|\mathcal{F}_t\right]$$
$$+\alpha_1E\left[\int_t^T e^{\beta s}|Z_s^{n+1}-Z_s^n|^2ds\middle|\mathcal{F}_t\right]+\alpha_2E\left[\int_t^T e^{\beta s}|X_s^{n+1}-X_s^n|^2ds\middle|\mathcal{F}_t\right].$$

Thus, by Burkholder-Davis-Gundy's inequality, with a positive constant c_1 and $\alpha_1=\frac{1}{2}$, $\alpha_2=1$, we have

$$E\left[\sup_{0\leq t\leq T}e^{\beta t}|Y_t^{n+1}-Y_t^n|^2\right]\leq c_1e^{\beta T}k_5^2E\left[|X_T^{n+1}-X_T^n|^2\right]$$
$$+c_1\frac{1}{2}E\left[\int_0^T e^{\beta s}|Z_s^{n+1}-Z_s^n|^2ds\right]+c_1E\left[\int_0^T e^{\beta s}|X_s^{n+1}-X_s^n|^2ds\right]$$
$$\leq 2c_1e^{\beta T}k_5^2E\left[|X_T^{n+1}-X_T^n|^2\right]+2c_1E\left[\int_0^T e^{\beta s}|X_s^{n+1}-X_s^n|^2ds\right].$$

It now follows from (4.2) that

$$E\left[\sup_{0\leq t\leq T}|Y_t^{n+1}-Y_t^n|^2\right]+E\left[\int_0^T(Z_s^{n+1}-Z_s^n)^2ds\right]$$
$$\leq 8(c_1+1)e^{\beta T}(k_5^2+T)T^2k_2^2E\left[\sup_{0\leq t\leq T}|Y_t^n-Y_t^{n-1}|^2\right].$$

Taking T small enough so that

$$8(c_1+1)e^{\beta T}(k_5^2+T)T^2k_2^2\leq\frac{1}{2},$$

we obtain that (X^n,Y^n,Z^n) is a Cauchy sequence in $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{S}^2(\mathbb{R}^l)\times\mathcal{H}^2(\mathbb{R}^{l\times d})$. Thus, it suffices to define C_2 by the conditions

$$\begin{cases}2T^2k_1^2\leq\frac{1}{2}\\ 8(c_1+1)e^{\beta T}(k_5^2+T)T^2k_2^2\leq\frac{1}{2}.\end{cases}$$

By continuity of b,g and h we have the existence of a solution (X,Y,Z) in $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{S}^2(\mathbb{R}^l)\times\mathcal{H}^2(\mathbb{R}^{l\times d})$ of FBSDE (1.1) and it follows from the boundedness of (Z^n) that $|Z_t|\leq M$. The uniqueness in $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{S}^2(\mathbb{R}^l)\times\mathcal{S}^\infty(\mathbb{R}^{l\times d})$ follows from the boundedness of Z and by repeating the above arguments on the difference of two solutions.

Step 3: If one of the functions b, g or h is not differentiable, we apply the technique of the proof of Theorem 3.1. Namely, we use an approximation by the smooth functions defined as follows: For $n \in \mathbb{N}$, let β_n^1, β_n^2 and β_n^3 be nonnegative C^∞ functions with support on $\{x \in \mathbb{R}^m : |x| \leq \frac{1}{n}\}$, $\{x \in \mathbb{R}^{m+l} : |x| \leq \frac{1}{n}\}$ and $\{x \in \mathbb{R}^{m+l+l\times d} : |x| \leq \frac{1}{n}\}$ respectively, and satisfying $\int_{\mathbb{R}^m} \beta_n^1(r)dr = 1$, $\int_{\mathbb{R}^{m+l}} \beta_n^2(r)dr = 1$ and $\int_{\mathbb{R}^{m+l+l\times d}} \beta_n^3(r)dr = 1$. We define the convolutions

$$b_t^n(x,y) := \int\limits_{\mathbb{R}^{m+l}} b_t(x',y')\beta_n^2(x'-x,y'-y)dx'dy', \quad h^n(x) := \int\limits_{\mathbb{R}^m} h(x')\beta_n^1(x'-x)dx',$$

$$g^n(u,x,y,z) := \int\limits_{\mathbb{R}^{m+l+l\times d}} g(u,x',y',z')\beta_n^3(x'-x,y'-y,z'-z)dx'dy'dz'.$$

It is easy to check that b^n satisfies (A1) with the constants k_1, k_2 and $2\lambda_1$ and that g^n and h^n satisfy (A4) - (A5) and (A3), respectively, with the same constants. From *Steps 1 and 2*, there exists a positive constant $\bar{C}$ independent of n such that if $T \leq \bar{C}$, FBSDE (1.1) with parameters (b^n, h^n, g^n) admits a unique solution $(X^n, Y^n, Z^n) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ and

$$|Z_t^n| \leq M.$$

By the Lipschitz continuity conditions on b and h and the locally Lipschitz condition of g, the sequences (b^n) and (h^n) converge uniformly to b and h on $\mathbb{R}^{m+l}$ and $\mathbb{R}^m$, respectively, and (g^n) converges to g uniformly on $\mathbb{R}^{m+l} \times \Lambda$ for any compact subset Λ of $\mathbb{R}^{l\times d}$. Combining these uniform convergences with the boundedness of Z^n, similar to above, there exists a constant $\tilde{C}$ depending on $k_1, k_2, k_3, k_4, k_5, \lambda_2, l$ and d such that if $T \leq \tilde{C}$, (X^n, Y^n, Z^n) is a Cauchy sequence in the Banach space $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{H}^2(\mathbb{R}^{l\times d})$.
In fact, for any $m, n \in \mathbb{N}$, using Cauchy-Schwarz' inequality we have

$$|X_t^n - X_t^m|^2 \leq T \int\limits_0^T |b_u^n(X_u^n, Y_u^n) - b_u^m(X_u^m, Y_u^m)|^2 \, du.$$

Thus, taking the supremum with respect to t and then expectation on both sides give

$$\begin{aligned}
&\|X^n - X^m\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\
&\quad \leq 3T \int\limits_0^T (|b_u^n(X_u^n, Y_u^n) - b_u(X_u^n, Y_u^n)|^2 + |b_u^m(X_u^m, Y_u^m) - b_u(X_u^m, Y_u^m)|^2 \\
&\qquad + |b_u(X_u^n, Y_u^n) - b_u(X_u^m, Y_u^m)|^2)du \\
&\quad \leq 3T \int\limits_0^T (|b_u^n(X_u^n, Y_u^n) - b_u(X_u^n, Y_u^n)|^2 + |b_u^m(X_u^m, Y_u^m) - b_u(X_u^m, Y_u^m)|^2) \, du \\
&\qquad + 3k_1^2T^2 \, \|X^n - X^m\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + 3k_2^2T^2 \, \|Y^n - Y^m\|^2_{\mathcal{S}^2(\mathbb{R}^l)} \qquad (4.4)
\end{aligned}$$

where the second inequality follows from (A1). On the other hand, applying Itô's formula as in Step 2, one has

$$
\begin{aligned}
&|Y_t^m - Y_t^n|^2 + \int_t^T |Z_u^n - Z_u^m|^2 \, du \\
&\leq |h^n(X_T^n) - h(X_T^n)|^2 + |h^m(X_T^m) - h(X_T^m)|^2 + k_5^2 \, |X_T^n - X_T^m|^2 \\
&\quad - 2 \int_t^T (Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n) dW_s \\
&\quad + \int_t^T |Y_u^n - Y_u^m| \, (|g_u^n(X_u^n, Y_u^n, Z_u^n) - g_u(X_u^n, Y_u^n, Z_u^n)| \\
&\qquad + |g_u^m(X_u^m, Y_u^m, Z_u^m) - g_u(X_u^m, Y_u^m, Z_u^m)| + k_3 \, |X_u^n - X_u^m| + k_4 \, |Y_u^n - Y_u^m| \\
&\qquad + \rho(M) \, |Z_u^n - Z_u^m|) du. \qquad (4.5)
\end{aligned}
$$

Taking expectation, due to Young's inequality we have

$$
\begin{aligned}
&\|Z^n - Z^m\|^2_{\mathcal{H}^2(\mathbb{R}^{l\times d})} \\
&\leq E[|h^n(X_T^n) - h(X_T^n)|^2] + E[|h^m(X_T^m) - h(X_T^m)|^2] + (k_5^2 + \frac{1}{2}Tk_3^2) \, \|X^n - X^m\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\
&\quad + \frac{1}{2}T \, \|Y^n - Y^m\|^2_{\mathcal{S}^2(\mathbb{R}^l)} + \frac{1}{2} \int_0^T |g_u^n(X_u^n, Y_u^n, Z_u^n) - g_u(X_u^n, Y_u^n, Z_u^n)|^2 \\
&\qquad + |g_u^m(X_u^m, Y_u^m, Z_u^m) - g_u(X_u^m, Y_u^m, Z_u^m)|^2 \, du \\
&\quad + \frac{1}{2}Tk_4^2\rho^2(M) \, \|Y^n - Y^m\|^2_{\mathcal{S}^2(\mathbb{R}^l)} + \frac{1}{2} \, \|Z^n - Z^m\|^2_{\mathcal{H}^2(\mathbb{R}^{l\times d})} .
\end{aligned}
$$

On the other hand, taking conditional expectation in (4.5) and then the supremum with respect to t and then expectation on both sides, we have due to Young's inequality

$$
\begin{aligned}
&\|Y^n - Y^m\|^2_{\mathcal{S}^2(\mathbb{R}^l)} \\
&\leq E[|h^n(X_T^n) - h(X_T^n)|^2] + E[|h^m(X_T^m) - h(X_T^m)|^2] + k_5^2 \, \|X^n - X^m\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\
&\quad + \frac{1}{2}T \, \|Y^n - Y^m\|^2_{\mathcal{S}^2(\mathbb{R}^l)} + \frac{1}{2} \int_0^T |g_u^n(X_u^n, Y_u^n, Z_u^n) - g_u(X_u^n, Y_u^n, Z_u^n)|^2 \\
&\qquad + |g_u^m(X_u^m, Y_u^m, Z_u^m) - g_u(X_u^m, Y_u^m, Z_u^m)|^2 \, du \\
&\quad + \frac{1}{2}Tk_3^2 \, \|X^n - X^m\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + \frac{1}{2}Tk_4^2\rho^2(M) \, \|Y^n - Y^m\|^2_{\mathcal{S}^2(\mathbb{R}^l)} + \frac{1}{2} \, \|Z^n - Z^m\|^2_{\mathcal{H}^2(\mathbb{R}^{l\times d})} .
\end{aligned}
$$

Combining (4.4) and (4.1) we observe that if T is small enough so that

$$
\begin{cases}
3k_1^2T^2 \leq \frac{1}{2} \\
\frac{1}{2}T + 3k_5^2k_2^2T^2 + \frac{3}{2}T^3k_3^2k_2^2 + \frac{1}{2}Tk_4^2\rho^2(M) \leq \frac{1}{2}
\end{cases}
$$

then, the uniform convergence of (b^n), g^n and (h^n) to b, g and h ensure that (X^n, Y^n, Z^n) is a Cauchy sequence. The verification that the limit (X, Y, Z) of the sequence (X^n, Y^n, Z^n) solves the FBSDE (1.1) uses continuity of the functions b, h and g, and that $|Z_t| \leq M$ is a consequence of the boundedness of (Z^n). Taking $C := \tilde{C} \wedge \bar{C}$ concludes the proof. □

4.2 Proof of Example 2.2

Let $k \geq 1$, $g(y,z) = y|z|^k$ and $R := \frac{\lambda_5 e^{TM^k}}{1-2M^k e^{TM^k}}$ with $T < \frac{1}{M^k}\log(\frac{1}{2M^k}) \wedge 1$. The function g restricted to the ball $\{y : |y| \leq R\} \times \{z : |z| \leq M\}$ is Lipschitz continuous, i.e. $|g(y,z) - g(y',z')| \leq M^k|y-y'| + 2RM^{k-1}|z-z'|$ for all $|y|, |y'| \leq R$ and $|z|, |z'| \leq M$. Thus, it can be extendent to a Lipchitz continuous function $\tilde{g}$ with the same Lipschitz constants on $\mathbb{R}^l \times \mathbb{R}^{l\times d}$. In particular, $\tilde{g}$ satisfies the conditions of Theorem 2.1. Thus, the FBSDE (2.2) with generator g replaced by $\tilde{g}$ admits a unique solution $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ with $|\tilde{Z}| \leq M$. Moreover, one has

$$|\tilde{Y}_t| \leq \lambda_5 + E\left[\int_t^T M^k|\tilde{Y}_s| + 2RM^{k-1}|\tilde{Z}_s|\, ds \,|\, \mathcal{F}_t\right]$$

so by Gronwall's inequality and boundedness of $\tilde{Z}$, it follows that

$$|\tilde{Y}_t| \leq (\lambda_5 + 2RM^k)e^{TM^k} = R.$$

That is, $\tilde{g}(\tilde{Y}_t, \tilde{Z}_t) = g(\tilde{Y}_t, \tilde{Z}_t)$, showing that $(\tilde{X}, \tilde{Y}, \tilde{Z})$ solves the FBSDE (2.2) with generator g. □

4.3 Proof of Remark 2.3

By construction, for T sufficiently small, there is a sequence $(X^n, Y^n, Z^n) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ converging in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{S}^2(\mathbb{R}^{l\times d})$ to the solution (X, Y, Z) of the FBSDE (1.1). Moreover, for each $t \in [0, T]$ it holds $(X_t^n, Y_t^n, Z_t^n) \in \mathcal{D}^{1,2}(\mathbb{R}^m) \times \mathcal{D}^{1,2}(\mathbb{R}^l) \times \mathcal{D}^{1,2}(\mathbb{R}^{l\times d})$ with $|D_r X_t^n| \leq 4\lambda_2$ and $|D_r Y_t^n| \leq M$ for all $r \in [0, T]$. For $t \in [0, T]$ one has

$$||X_t^n - X_t||_{L^2} + ||Y_t^n - Y_t||_{L^2} \to 0.$$

Thus, it follows from [27, Proposition 1.2.3] that $(X_t, Y_t) \in \mathcal{D}^{1,2}(\mathbb{R}^m) \times \mathcal{D}^{1,2}(\mathbb{R}^l)$. □

4.4 Proof of Theorem 2.5

First assume that (A6) is satisfied. If $T \leq C$, then the result follows from Theorem 2.1.
Assume $T > C$ and let $\tilde{h}_{\bar{M}} : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function whose derivative is bounded by 1 and such that $\tilde{h}'_{\bar{M}}(a) = 1$ for all $-\bar{M} \leq a \leq \bar{M}$ and

$$\tilde{h}_{\bar{M}}(a) = \begin{cases} (\bar{M}+1) & \text{if} \quad a > \bar{M}+2 \\ a & \text{if} \quad |a| \leq \bar{M} \\ -(\bar{M}+1) & \text{if} \quad a < -(\bar{M}+2). \end{cases}$$

An example of such a function is given by

$$\tilde{h}_{\bar{M}}(a) = \begin{cases} \left(-\bar{M}^2 + 2\bar{M}a - a(a-4)\right)/4 & \text{if} \quad a \in [\bar{M}, \bar{M}+2] \\ \left(\bar{M}^2 + 2\bar{M}a + a(a+4)\right)/4 & \text{if} \quad [-(\bar{M}+2), -\bar{M}], \end{cases}$$

see Imkeller and dos Reis [19]. By the assumptions (A3) the function $\tilde{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \to \mathbb{R}^l$ defined by

$$\tilde{g}_t(x, y, z) := g_t(x, y, h_{\bar{M}}(z)) \tag{4.6}$$

with $h_{\bar{M}}(z) := (\tilde{h}_{\bar{M}}(z^{ij}))_{ij}$ being Lipschitz continuous in all variables. Thus, it follows from Delarue [8, Theorem 2.6] that the equation

$$\begin{cases} \tilde{X}_t = x + \int_0^t b_u(\tilde{X}_u, \tilde{Y}_u)\, du + \int_0^t \sigma_u \, dW_u \\ \tilde{Y}_t = h(\tilde{X}_T) + \int_t^T \tilde{g}_u(\tilde{X}_u, \tilde{Y}_u, \tilde{Z}_u)\, du - \int_t^T \tilde{Z}_u \, dW_u, \quad t \in [0,T] \end{cases} \tag{4.7}$$

admits a unique solution $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$. Moreover, there exists a Lipschitz continuous function $\theta : [0,T] \times \mathbb{R}^m \to \mathbb{R}^l$ bounded by K_5 such that $\tilde{Y}_t = \theta(t, \tilde{X}_t)$ for all $t \in [0,T]$. In fact, for every $x, x' \in \mathbb{R}^d$, $t \in [0,T]$ and $i = 1, \dots, l$, it follows by Itô's formula that

$$\begin{aligned}
&|\theta(t, \tilde{X}_t^x) - \theta(t, \tilde{X}_t^{x'})|^2 \\
&= |h(\tilde{X}_T^x) - h(\tilde{X}_T^{x'})|^2 - \int_t^T 2 \sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}\right) dW_u \\
&\quad + \int_t^T 2 \sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'})\right) du \\
&\le |h(\tilde{X}_T^x) - h(\tilde{X}_T^{x'})|^2 - \int_t^T 2 \sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}\right) dW_u \\
&\quad + \int_t^T 2|\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})| \left(k_3|\tilde{X}_u^x - \tilde{X}_u^{x'}| + \rho(\bar{M}\sqrt{dl})|\tilde{Z}_u^x - \tilde{Z}_u^{x'}|\right) du \\
&\quad - \int_t^T 2K_4|\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})|^2 + |\tilde{Z}_u^x - \tilde{Z}_u^{x'}|^2 du
\end{aligned}$$

$$\begin{aligned}
&\le |h(\tilde{X}_T^x) - h(\tilde{X}_T^{x'})|^2 - \int_t^T 2 \sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}\right) dW_u \\
&\quad + \int_t^T k_3|\tilde{X}_u^x - \tilde{X}_u^{x'}|^2 - \left(2K_4 - k_3 - \rho^2(\bar{M}\sqrt{dl})\right)|\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})|^2 du
\end{aligned}$$

Since by Gronwall's lemma we have

$$|\tilde{X}_s^x - \tilde{X}_s^{x'}| \leq (|\tilde{X}_t^x - \tilde{X}_t^{x'}| + k_2 \int_t^s |\tilde{Y}_u^x - \tilde{Y}_u^{x'}| \, du) e^{k_1 T}, \quad s \in [t, T],$$

it holds

$$\begin{aligned}
&\left|\theta(t, \tilde{X}_t^x) - \theta(t, \tilde{X}_t^{x'})\right|^2 \\
&\leq E\bigg[2e^{2k_1T}(k_5^2 + k_3 T)|\tilde{X}_t^x - \tilde{X}_t^{x'}|^2 \\
&\qquad + \Big[2e^{2k_1T}k_2^2 T(k_5^2 + k_3 T) + k_3 + \rho^2(\bar{M}\sqrt{dl}) - K_4\Big] \int_t^T |\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})|^2 \, du \,|\, \mathcal{F}_t\bigg] \\
&\leq 2e^{2k_1T}(k_5^2 + k_3 T)|\tilde{X}_t^x - \tilde{X}_t^{x'}|^2.
\end{aligned}$$

Thus,

$$\left|\theta(t, \tilde{X}_t^x) - \theta(t, \tilde{X}_t^{x'})\right| \leq K_5 |\tilde{X}_t^x - \tilde{X}_t^{x'}|.$$

Let $\bar{C} := C(k_1, k_2, k_3, k_4, K_5, \lambda_2, l, d)$ and put $N = \lfloor T/\bar{C} \rfloor$, where $\lfloor a \rfloor$ denotes the integer part of a, and $t_i := i\bar{C}$, $i = 0, \ldots, N$ and $t_{N+1} = T$. Since $t_1 \leq \bar{C}$, by Theorem 2.1 the FBSDE

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u) \, du + \int_0^t \sigma_u \, dW_u \\ Y_t = \theta(t_1, X_{t_1}) + \int_t^{t_1} g_u(X_u, Y_u, Z_u) \, du - \int_t^{t_1} Z_u \, dW_u, \quad t \in [0, t_1] \end{cases}$$

admits a unique solution (X^1, Y^1, Z^1) such that $\left|Z_t^1\right| \leq \bar{M}$ with $\bar{M} = 8\lambda_2 K_5 \sqrt{dl}$ for all $t \in [0, t_1]$. Therefore, $(X^1, Y^1, Z^1)1_{[0,t_1]} = (\tilde{X}, \tilde{Y}, \tilde{Z})1_{[0,t_1]}$. Similarly, we obtain a family (X^i, Y^i, Z^i) of solutions of the FBSDEs

$$\begin{cases} X_t = \tilde{X}_{t_{i-1}} + \int_{t_{i-1}}^t b_u(X_u, Y_u) \, du + \int_{t_{i-1}}^t \sigma_u \, dW_u \\ Y_t = \theta(t_i, X_{t_i}) + \int_t^{t_i} g_u(X_u, Y_u, Z_u) \, du - \int_t^{t_i} Z_u \, dW_u, \quad t \in [t_{i-1}, t_i] \end{cases}$$

such that $(X^i, Y^i, Z^i)1_{[t_{i-1},t_i]} = (\tilde{X}, \tilde{Y}, \tilde{Z})1_{[t_{i-1},t_i]}$, $i = 1, \ldots, N+1$. Define

$$X := \sum_{i=1}^{N+1} X^i 1_{[t_{i-1},t_i]}, \quad Y := \sum_{i=1}^{N+1} Y^i 1_{[t_{i-1},t_i]} \quad \text{and} \quad Z := \sum_{i=1}^{N+1} Z^i 1_{[t_{i-1},t_i]}.$$

Then, $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ is the unique solution of the FBSDE (1.1) satisfying $|Z_t| \leq \bar{M}$ for all $t \in [0, T]$. In fact, it is clear that $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$ as a finite sum of elements of the same space. Let $t \in [0, T]$ and $i = 1, \ldots, N+1$ such that $t \in [t_{i-1}, t_i]$. We have

$$\begin{aligned}
x + \int_0^t b_u(X_u) \, du + \int_0^t \sigma_u \, du &= x + \sum_{j=1}^{i} \left(\int_{t_{j-1}}^{t_j \wedge t} b_u(X_u^j) \, du + \int_{t_{j-1}}^{t_j \wedge t} \sigma_u \, dW_u \right) \\
&= X_t^i = X_t
\end{aligned}$$

and

$$h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u)\,du - \int_t^T Z_u\,dW_u$$
$$= h(X_T^{N+1}) + \sum_{j=i}^{N+1} \left(\int_{t_{j-1}\vee t}^{t_j} g_u(X_u^j, Y_u^j, Z_u^j)\,du - \int_{t_{j-1}\vee t}^{t_j} Z_u^j\,dW_u \right) = Y_t^i = Y_t.$$

That is, (X, Y, Z) satisfies Equation (1.1). □

In the case where (A7) is satisfied, the proof is similar and we only need to provide the argument for the Lipschitz continuity of θ. If $T \le C$, then the result follows from Theorem 2.1.

Assume $T > C$ and let $\tilde{h}_M : \mathbb{R} \to \mathbb{R}$ be a suitable truncating function and $\tilde{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \to \mathbb{R}^l$ defined by

$$\tilde{g}_t(x, y, z) := g_t(x, y, h_M(z)) \tag{4.8}$$

with $h_M(z) := (\tilde{h}_M(z^{ij}))_{ij}$. It follows from Delarue [8, Theorem 2.6] that the equation

$$\begin{cases} \tilde{X}_t = x + \int_0^t b_u(\tilde{X}_u, \tilde{Y}_u)\,du + \int_0^t \sigma_u\,dW_u \\ \tilde{Y}_t = h(\tilde{X}_T) + \int_t^T \tilde{g}_u(\tilde{X}_u, \tilde{Y}_u, \tilde{Z}_u)\,du - \int_t^T \tilde{Z}_u\,dW_u, \quad t \in [0,T] \end{cases} \tag{4.9}$$

admits a unique solution $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^l) \times \mathcal{S}^\infty(\mathbb{R}^{l\times d})$. Moreover, there exists a Lipschitz continuous function $\theta : [0,T] \times \mathbb{R}^m \to \mathbb{R}^l$ bounded by k_5 such that $\tilde{Y}_t = \theta(t, \tilde{X}_t)$ for all $t \in [0,T]$. In fact, for every $x, x' \in \mathbb{R}^d$, $t \in [0,T]$ and $i = 1, \dots, l$ applying Itô's formula we have

$$\begin{aligned}
&|\theta(t, \tilde{X}_t^x) - \theta(t, \tilde{X}_t^{x'})|^2 \\
&= |h(\tilde{X}_T^x) - h(\tilde{X}_T^{x'})|^2 - \int_t^T 2\sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}\right) dW_u \\
&\quad + \int_t^T 2\sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(g_u^i(\tilde{X}_u^x, \tilde{Y}_u^x, \tilde{Z}_u^x) - g_u^i(\tilde{X}_u^{x'}, \tilde{Y}_u^{x'}, \tilde{Z}_u^{x'})\right) du \\
&\le |h(\tilde{X}_T^x) - h(\tilde{X}_T^{x'})|^2 - \int_t^T 2\sum_{i=1}^l \left(\theta^i(u, \tilde{X}_u^x) - \theta^i(u, \tilde{X}_u^{x'})\right)\left(\tilde{Z}_u^{x,i} - \tilde{Z}_u^{x',i}\right) dW_u \\
&\quad + \int_t^T 2|\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})| \left(k_3|\tilde{X}_u^x - \tilde{X}_u^{x'}| + \rho(M\sqrt{ld})|\tilde{Z}_u^x - \tilde{Z}_u^{x'}|\right) du \\
&\quad - \int_t^T 2K_4|\theta(u, \tilde{X}_u^x) - \theta(u, \tilde{X}_u^{x'})|^2 + |\tilde{Z}_u^x - \tilde{Z}_u^{x'}|^2 du
\end{aligned}$$

$$\leq |h(\tilde{X}^x_T) - h(\tilde{X}^{x'}_T)|^2 - \int_t^T 2\sum_{i=1}^{l}\left(\theta^i(u,\tilde{X}^x_u) - \theta^i(u,\tilde{X}^{x'}_u)\right)\left(\tilde{Z}^{x,i}_u - \tilde{Z}^{x',i}_u\right) dW_u$$

$$+ \int_t^T 2k_3|\theta(u,\tilde{X}^x_u) - \theta(u,\tilde{X}^{x'}_u)||\tilde{X}^x_u - \tilde{X}^{x'}_u| - \left(K_4 - \rho^2(M\sqrt{ld})\right)|\theta(u,\tilde{X}^x_u) - \theta(u,\tilde{X}^{x'}_u)|^2 du$$

Since for $s \in [t,T]$,

$$|\tilde{X}^x_s - \tilde{X}^{x'}_s|^2 \leq |\tilde{X}^x_t - \tilde{X}^{x'}_t|^2 + \int_t^s -2K_1|\tilde{X}^x_u - \tilde{X}^{x'}_u|^2 + 2k_2|\tilde{X}^x_u - \tilde{X}^{x'}_u||\tilde{Y}^x_u - \tilde{Y}^{x'}_u|\,du,$$

it holds

$$\begin{aligned}
&\left|\theta(t,\tilde{X}^x_t) - \theta(t,\tilde{X}^{x'}_t)\right|^2 \\
&\leq E\Bigg[k_5^2|\tilde{X}^x_t - \tilde{X}^{x'}_t|^2 + \int_t^s -2K_1k_5^2|\tilde{X}^x_u - \tilde{X}^{x'}_u|^2 + (2k_2k_5^2 + 2k_3)|\tilde{X}^x_u - \tilde{X}^{x'}_u||\tilde{Y}^x_u - \tilde{Y}^{x'}_u|\,du \\
&\qquad + \left(\rho^2(M\sqrt{ld}) - K_4\right)\int_t^T |\theta(u,\tilde{X}^x_u) - \theta(u,\tilde{X}^{x'}_u)|\,du \,|\, \mathcal{F}_t\Bigg] \\
&\leq k_5^2|\tilde{X}^x_t - \tilde{X}^{x'}_t|^2.
\end{aligned}$$

Thus,

$$\left|\theta(t,\tilde{X}^x_t) - \theta(t,\tilde{X}^{x'}_t)\right| \leq k_5|\tilde{X}^x_t - \tilde{X}^{x'}_t|.$$

Having proved this Lipschitz continuity property, the rest of the proof is exactly the same as in the first part. □

Remark 4.1. With the techniques presented above, it is hard to consider systems where the drift b depends on z, since we do not have good enough estimates on the Malliavin derivative of the control process Z. Similarly, when the diffusion coefficient σ is a function of x, y or z, we loose the estimates on the Malliavin derivatives of the solutions. These cases (even the in non-Markovian case) can nevertheless be studied under stronger assumptions, we refer to [23] for details. ♦

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