Asymptotically stable dynamic risk assessments

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Summary: In this paper we study asymptotically stable risk assessments (or equivalently risk measures) which have the property that an unacceptable position cannot become acceptable by adding a huge cash-flow far in the future. Under an additional continuity assumption, these risk assessments are exactly those which have a a robust representation in terms of test probabilities that are supported on a finite time interval. For time-consistent risk assessments we give conditions on their generators which guarantee asymptotic stability.

1 Introduction

The starting point of this paper is the investigation of risk assessments which are not influenced by statements like: "Far in the future I shall be extremely rich". For such risk assessments negative values in the near future cannot be compensated by extremely positive values far away. We call such assessments *asymptotically stable*.

Risk assessments (or equivalently risk measures) have been widely studied since the lighthouse paper [3] of Artzner et al. in 1999; see also [7] and [8]. Our focus is on risk assessments of stochastic processes; see for instance [1], [2], [4], [5], [6], [10], [11], [13], [14]. Since we are interested in an asymptotic concept we consider risk assessments for discrete time stochastic processes with an infinite time horizon. In the first main result of this paper we characterize asymptotic stable risk assessments under the additional assumption of local continuity from below by a robust representation. It turns out that the test probabilities of this representation are exactly the so-called local probabilities whose support is restricted to a finite time interval. This is equivalent to the fact that the acceptance set of the risk assessment is closed in the weak topology induced by local test probabilities.

The main concept of this paper is asymptotic stability which is defined in terms of the acceptance set of the risk assessment: If a position modeled as a stochastic process X is not acceptable, then there exists a time horizon T such that $X\mathbb{1}_{(0,T)} + N\mathbb{1}_{[T,\infty)}$ is still not acceptable regardless of the size of the future position N. In that sense, asymptotic stable risk assessments may be regarded as *immune* against extremely high values in the far future.

Using the property of time-consistency, we show in the second part of the paper how risk assessments can be constructed by composing a sequence of generators. We provide

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a condition on the generators such that the resulting risk assessment is asymptotically stable, and discuss several examples.

The structure of this paper is as follows. In Section 2 we give the main notation and definitions. The robust representation of asymptotically stable risk assessments is presented in the Section 3. In Section 4 we construct dynamic risk assessments which are asymptotically safe and finally give some examples in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space. Denote by L^p (resp. L^p_t) the space of all $\mathcal{F}$-measurable (resp. $\mathcal{F}_t$-measurable) random variables with finite pth moment if $p \in [1, \infty)$, and which are essentially bounded if $p = \infty$, and where two of them are identified if they agree $\mathbb{P}$-almost surely. Let $\mathcal{R}^\infty$ be the space of all adapted processes $X : \Omega \times \mathbb{N} \to \mathbb{R}$ such that $X^* := \sup_{t\in\mathbb{N}} |X_t| \in L^\infty$. On $\mathcal{R}^\infty$ we work with the partial order $X \geq Y$ whenever $X_t \geq Y_t$ for all $t \in \mathbb{N}$.

Definition 2.1 *A concave risk assessment on $\mathcal{R}^\infty$ is a function $\phi : \mathcal{R}^\infty \to \mathbb{R}$ which satisfies*

(i) $\phi(0) = 0$,

(ii) $\phi(X + m\mathbb{I}_{[1,\infty)}) = \phi(X) + m$ *for all* $m \in \mathbb{R}$,

(iii) $\phi(X) \geq \phi(Y)$ *whenever* $X \geq Y$,

(iv) $\phi(\lambda X + (1-\lambda)Y) \geq \lambda\phi(X) + (1-\lambda)\phi(Y)$ *for all* $\lambda \in (0, 1)$,

for all $X, Y \in \mathcal{R}^\infty$.

If in addition to (i) – (iv), ϕ is positively homogeneous, i.e. $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \geq 0$ then ϕ is called coherent risk assessment.

The acceptance set of a risk assessment ϕ is defined as $\mathcal{C} := \{X \in \mathcal{R}^\infty : \phi(X) \geq 0\}$. In line with [5] for two stopping times τ and θ such that $0 \leq \tau \leq \theta < \infty$ we define the projection $\pi_{\tau,\theta} : \mathcal{R}^\infty \to \mathcal{R}^\infty$ by

$$\pi_{\tau,\theta}(X)_t := \mathbb{I}_{\{\tau\leq t\}} X_{t\wedge\theta}, \quad t \in \mathbb{N}.$$

Definition 2.2 *A risk assessment $\phi : \mathcal{R}^\infty \to \mathbb{R}$ is locally continuous from below, if for every sequence (X^n) in $\mathcal{R}^\infty$ and $X \in \mathcal{R}^\infty$ such that $X^n \leq X^{n+1}$ and $X^n_t \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$, one has*

$$\phi(\pi_{0,T}(X^n)) \to \phi(\pi_{0,T}(X))$$

for all $T \in \mathbb{N}$.

Let $\mathcal{A}^1$ be the space of all adapted processes $a : \Omega \times \mathbb{N} \to \mathbb{R}$ such that $\sum_{t\in\mathbb{N}} |\Delta a_t| \in L^1$, where $\Delta a_t := a_t - a_{t-1}$ with the convention $a_0 := 0$. Further, denote by $\mathcal{A}^{1,\mathrm{loc}}$ the set of those $a \in \mathcal{A}^1$ which are eventually constant, that is $a_t = a_T$ for all $t \geq T$ for some $T \in \mathbb{N}$. The linear spaces $\mathcal{R}^\infty$ and $\mathcal{A}^1$ are in duality by the dual pairing

$$\langle X, a \rangle := \mathbb{E}\left[\sum_{t\in\mathbb{N}} X_t \Delta a_t\right].$$

Note that for $a \in \mathcal{A}^{1,\mathrm{loc}}$ one has $\langle X, a\rangle := \sum_{t=1}^{T} \mathbb{E}\left[X_t \Delta a_t\right]$ for some $T \in \mathbb{N}$.

For any time horizon $T \in \mathbb{N}$ we define the subspaces

$$\mathcal{R}^\infty_{0,T} := \pi_{0,T}\mathcal{R}^\infty \quad \text{and} \quad \mathcal{A}^1_{0,T} := \pi_{0,T}\mathcal{A}^1. \tag{2.1}$$

Finally, let $\mathcal{A}^1_+$ denote the set of those $a \in \mathcal{A}^1$ which satisfy $a_t \leq a_{t+1}$ for all $t \in \mathbb{N}$, and $\mathcal{A}^1_1$ the set of those $a \in \mathcal{A}^1_+$ for which $\mathbb{E}\left[\sum_{t\in\mathbb{N}} \Delta a_t\right] = 1$. Similarly, we define $\mathcal{A}^{1,\mathrm{loc}}_1 := \mathcal{A}^{1,\mathrm{loc}} \cap \mathcal{A}^1_1$. Elements of $\mathcal{A}^{1,\mathrm{loc}}_1$ are referred to as *local test probabilities*.

3 Asymptotically stable risk assessments

The focus of this paper is to characterize risk assessments on $\mathcal{R}^\infty$ which have the desired property that an unacceptable position cannot become acceptable by adding a huge cash-flow far in the future.

Definition 3.1 *A concave risk assessment ϕ on $\mathcal{R}^\infty$ is called* asymptotically stable *if for each $X \in \mathcal{R}^\infty$ with $X \notin \mathcal{C}$ there exists a time horizon $T \in \mathbb{N}$ such that*

$$X\mathbb{1}_{(0,T)} + N\mathbb{1}_{[T,\infty)} \notin \mathcal{C}$$

for all $N \in \mathbb{N}$.

Although asymptotically stable risk assessments neglect asymptotic benefits, they may take into account asymptotic losses. Asymptotic stability can be characterized as follows:

Proposition 3.2 *For a concave risk assessment ϕ on $\mathcal{R}^\infty$ the following are equivalent:*

(i) ϕ is asymptotically stable

(ii) $X \in \mathcal{C}$ iff for every $t \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)} \in \mathcal{C}$

(iii) For every $X \in \mathcal{R}^\infty$, $\gamma_t(X) := \sup_{N\in\mathbb{N}}\left[\phi(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}) - \phi(X)\right] \to 0$ as $t \to \infty$.

Proof: We only show that (i) and (iii) are equivalent, the equivalence between (i) and (ii) is obvious. To that end, suppose that (i) holds, but $\sup_{t\in\mathbb{N}} \gamma_t(X) \geq \varepsilon$ for some $\varepsilon > 0$. By (ii) of Definition 2.1 we may assume that $\phi(X) = -\varepsilon/2$ so that $X \notin \mathcal{C}$. Hence, for any $t \in \mathbb{N}$ one has

$$\sup_{N\in\mathbb{N}} \phi(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}) = \gamma_t(X) + \phi(X) \geq \frac{\varepsilon}{2},$$

so that $X1\!\!1_{(0,t)} + N1\!\!1_{[t,\infty)} \in \mathcal{C}$ for all $t \in \mathbb{N}$ and sufficiently large $N \in \mathbb{N}$. Hence (i) implies $X \in \mathcal{C}$, which is a contradiction, so that (iii) has to hold.

Conversely, suppose that (iii) holds and $X \notin \mathcal{C}$. Since

$$\sup_{N\in\mathbb{N}} \phi(X1\!\!1_{(0,t)} + N1\!\!1_{[t,\infty)}) = \gamma_t(X) + \phi(X), \quad \text{for all } t \in \mathbb{N},$$

$\phi(X) < 0$ and $\gamma_t(X) \to 0$, it follows that $\sup_{N\in\mathbb{N}} \phi(X1\!\!1_{(0,t_0)} + N1\!\!1_{[t_0,\infty)}) < 0$ for some t_0 large enough. Hence $X1\!\!1_{(0,t_0)} + N1\!\!1_{(t_0,\infty)} \notin \mathcal{C}$ for all $N \in \mathbb{N}$, which shows (i). □

In a next step we give the robust representation of asymptotically stable concave risk assessments. The general representation theory provides formulas of the form

$$\phi(X) = \inf_{a\in\mathscr{A}} \{\langle X, a\rangle - \phi^*(a)\}, \quad \text{for all } X \in \mathcal{R}^\infty, \tag{3.1}$$

where the 'dual set' $\mathscr{A}$ is a convex set of linear forms on $\mathcal{R}^\infty$ and the conjugate function ϕ^* is given by

$$\phi^*(a) := \inf_{X\in\mathcal{R}^\infty} \{\langle X, a\rangle - \phi(X)\} = \inf_{X\in\mathcal{C}} \langle X, a\rangle, \quad \text{for all } a \in \mathscr{A}, \tag{3.2}$$

and takes values in $[-\infty, 0]$. The second equality in (3.2) follows from

$$\inf_{X\in\mathcal{C}} \langle X, a\rangle \geq \inf_{X\in\mathcal{C}} \{\langle X - \phi(X), a\rangle\} \geq \inf_{X\in\mathcal{R}^\infty} \{\langle X - \phi(X), a\rangle\} \geq \inf_{X'\in\mathcal{C}} \langle X', a\rangle$$

because $X - \phi(X) \in \mathcal{C}$.

In our context, the robust representation of ϕ is given in the following theorem which is our main result in the static case. It characterizes the property of asymptotical stability for risk assessments which are locally continuous from below in terms of local test probabilities.

Theorem 3.3 *Let $\phi : \mathcal{R}^\infty \to \mathbb{R}$ be a concave risk assessment which is locally continuous from below. The following statements are equivalent:*

(i) *ϕ is asymptotically stable*

(ii) *The acceptance set $\mathcal{C}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^{1,\mathrm{loc}})$-closed*

(iii) *ϕ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^{1,\mathrm{loc}})$-upper semicontinuous*

(iv) *ϕ has a robust representation with local test probabilities:*

$$\phi(X) = \inf_{a\in\mathcal{A}_1^{1,\mathrm{loc}}} \{\langle X, a\rangle - \phi^*(a)\}, \quad \text{for all } X \in \mathcal{R}^\infty \tag{3.3}$$

(v) *For any sequence (X^n) and X in $\mathcal{R}^\infty$ such that $\sup_{n\in\mathbb{N}} |X_t^n| \in L^\infty$ and $X_t^n \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$, one has $\phi(X) \geq \limsup_{n\to\infty} \phi(X^n)$.*

Proof: $(i) \Rightarrow (ii)$: We have to show that

$$X \in \mathcal{C} \quad \Longleftrightarrow \quad \langle X, a\rangle - \phi^*(a) \geq 0 \quad \text{for all } a \in \mathcal{A}_1^{1,\mathrm{loc}}. \tag{3.4}$$

That the left hand side implies the right hand side follows directly from the definition of ϕ^*. The converse implication is shown in the following six steps.
Step 1: Suppose $X \notin \mathcal{C}$. By (ii) of Definition 2.1 there exists $\varepsilon \in (0,1]$ such that $X + \varepsilon \notin \mathcal{C}$. Since ϕ is asymptotically stable there exists $t \in \mathbb{N}$ such that

$$(X+\varepsilon)\mathbb{1}_{(0,t)} + (N+1)\mathbb{1}_{[t,\infty)} \notin \mathcal{C} \quad \text{for all } N \in \mathbb{N}. \tag{3.5}$$

Step 2: Here we make a deviation to use the duality between $L^\infty(\Omega', \mathcal{F}')$ and the set $\mathcal{M}^f(\Omega', \mathcal{F}')$ of finitely additive measures on some measurable space $(\Omega', \mathcal{F}')$. We follow [1], [2], [5], or [11].
Set $\Omega' := \Omega \times \mathbb{N}$, $\mathcal{F}' := \sigma\{\mathcal{F}_n \times \{n\} : n \in \mathbb{N}\}$, $\mu^0(B) := \sum_{n\geq 1} 2^{-n}\mathbb{P}(B_{|n})$ where $B_{|n} = \{\omega \in \Omega : (\omega, n) \in B\}$ for $B \in \mathcal{F}'$. μ^0 is a probability measure on $(\Omega', \mathcal{F}')$.
Further, let $\mathcal{M}_1^f$ the set of positive finitely additive measures μ on $(\Omega', \mathcal{F}')$ with $\mu(\Omega') = 1$ and $\mu(B \times \{n\}) = 0$ for all $B \in \mathcal{F}_n$ with $\mathbb{P}[B] = 0$. Identifying $X \in \mathcal{R}^\infty$ with $X'(\omega, n) = X_n(\omega) \in L^\infty(\Omega', \mathcal{F}')$ and writing $\langle X, \mu\rangle$ instead of $\langle X', \mu\rangle$, we get for $\phi'(X') = \phi(X)$ the representation (see [12])

$$\phi(X) = \phi'(X') = \min_{\mu \in \mathcal{M}_1^f} \{\langle X, \mu\rangle - \phi'^*(\mu)\}.$$

where again $\phi'^*(\mu) := \inf_{X \in \mathcal{C}} \langle X', \mu\rangle$ takes values in $[-\infty, 0]$.
Now the statement (3.5) implies the existence of a sequence (μ^N) in $\mathcal{M}_1^f$ with

$$\left\langle (X+\varepsilon)\mathbb{1}_{(0,t)} + (N+1)\mathbb{1}_{[t,\infty)}, \mu^N \right\rangle - \phi'^*(\mu^N) < 0 \quad \text{for all } N \in \mathbb{N}. \tag{3.6}$$

Since $\phi'^*(\mu^N) \leq 0$, it follows that

$$\left\langle \mathbb{1}_{\Omega\times[t,\infty)}, \mu^N \right\rangle \leq \frac{\|X\|_{\mathcal{R}^\infty} + \varepsilon}{N+1} \quad \text{and} \quad 0 \geq \phi'^*(\mu^N) \geq -\|X\|_{\mathcal{R}^\infty} \quad \text{for all } N \in \mathbb{N}. \tag{3.7}$$

Step 3: Next we show that the sequence (μ^N) satisfies:

$$\begin{aligned} &\forall\, \eta > 0, \forall\, B_\kappa \in \mathcal{F}', B_\kappa \nearrow \Omega' \text{ for } \kappa \to \infty\ \exists\, N_0, \kappa_0 \in \mathbb{N}, \text{ such that} \\ &\left\langle \mathbb{1}_{B_\kappa}, \mu^N \right\rangle \geq 1 - \eta \text{ for all } N \geq N_0 \text{ and } \kappa \geq \kappa_0. \end{aligned} \tag{3.8}$$

Let $\eta > 0$ and $B_\kappa \in \mathcal{F}'$ be an increasing sequence of subsets of Ω' with $B_\kappa \nearrow \Omega'$ for $\kappa \to \infty$. By (3.7) there exists $N_0 \in \mathbb{N}$ such that $\left\langle \mathbb{1}_{\Omega\times[t,\infty)}, \mu^N \right\rangle \leq \eta$ for all $N \geq N_0$. Fix $M \in \mathbb{N}$ such that $M \geq \|X\|_{\mathcal{R}^\infty} / \eta$. By the local continuity from below of ϕ we get

$$\begin{aligned} &\lim_{\kappa\to\infty} \inf_{N \geq N_0} \left\{ \left\langle M\mathbb{1}_{B_\kappa}, \mu^N \right\rangle - \phi'^*(\mu^N) \right\} \\ &\geq \lim_{\kappa\to\infty} \inf_{N\geq N_0} \left\{ \left\langle M(\mathbb{1}_{B_\kappa \cap \Omega\times(0,t)} + \mathbb{1}_{\Omega\times[t,\infty)}), \mu^N \right\rangle - \phi'^*(\mu^N) \right\} - M\eta \\ &\geq \lim_{\kappa\to\infty} \phi'(M(\mathbb{1}_{B_\kappa\cap\Omega\times(0,t)} + \mathbb{1}_{\Omega\times[t,\infty)})) - M\eta \\ &= \lim_{\kappa\to\infty} \phi\left(M(\mathbb{1}_{B_\kappa\cap\Omega\times\{1\}}, \ldots, \mathbb{1}_{B_\kappa\cap\Omega\times\{t-1\}}, \mathbb{1}_{\Omega\times\{t\}}, \ldots)\right) - M\eta \\ &= \phi(M\mathbb{1}_{(0,\infty)}) - M\eta = M(1-\eta). \end{aligned}$$

With the second inequality in (3.7), it follows that

$$1 \geq \lim_{\kappa\to\infty} \inf_{N\geq N_0} \left\langle 1\!\!1_{B_\kappa}, \mu^N \right\rangle \geq 1 - \eta - \|X\|_{\mathcal{R}^\infty} / M \geq 1 - 2\eta$$

This shows (3.8).
Step 4: In this step we prove

$$\forall\, \eta > 0,\ \exists\, \delta > 0 \text{ and } N \in \mathbb{N},\ \text{so that } \forall D \in \mathcal{F}' \text{ with } \sup_{0\leq N'\leq N} \left\langle 1\!\!1_D, \mu^{N'} \right\rangle < \delta$$
$$\text{implies } \left\langle 1\!\!1_D, \mu^M \right\rangle < \eta \text{ for all } M \in \mathbb{N}. \tag{3.9}$$

Assume that (3.9) does not hold. Then there exists $\eta > 0$ such that for all $\delta_N = \eta/2^N$ there exists $D_N \in \mathcal{F}'$ with $\sup_{0\leq N'\leq N} \left\langle 1\!\!1_{D_N}, \mu^{N'} \right\rangle < \eta/2^N$, but $\left\langle 1\!\!1_{D_N}, \mu^M \right\rangle \geq \eta$ for some $M \in \mathbb{N}$. We set $B_\kappa := \Omega' \setminus \bigcup_{N\geq\kappa} D_N$ such that $\mu^0(B_\kappa) \geq 1 - 2\eta/2^\kappa$ or $B_\kappa \nearrow \Omega'$ for $\kappa \to \infty$. (Here we use the fact that μ^0 is σ-additive, while μ^N are only finitely additive for for $N \geq 1$). By (3.8) there exist $N_0, \kappa_0 \in \mathbb{N}$ such that $\left\langle 1\!\!1_{B_\kappa}, \mu^M \right\rangle \geq 1 - \eta/2$ for all $M \geq N_0$ and $\kappa \geq \kappa_0$. Now for $\kappa' = \max(N_0, \kappa_0)$ we find not only $\sup_{0\leq N'\leq\kappa'} \left\langle 1\!\!1_{B^c_{\kappa'}}, \mu^{N'} \right\rangle < \eta/2^{\kappa'} \leq \eta/2$, but also $\left\langle 1\!\!1_{B^c_{\kappa'}}, \mu^M \right\rangle \leq \eta/2$ for all $M \geq \kappa'$, contradicting the fact that we have $\left\langle 1\!\!1_{B^c_{\kappa'}}, \mu^{M_{\kappa'}} \right\rangle \geq \eta$ for $M_{\kappa'} \in \mathbb{N}$. Thus assertion (3.9) is shown.
Step 5: We define $\overline{\mu} := \sum_{N'\geq 0} \mu^{N'}/2^{N'}$ and conclude for all $\eta > 0$ using $\delta > 0$ and N from (3.9): For any $B' \in \mathcal{F}'$ with $\langle 1\!\!1_{B'}, \overline{\mu} \rangle < \delta/2^N$ we have $\left\langle 1\!\!1_{B'}, \mu^{N'} \right\rangle < \delta$ for all $0 \leq N' \leq N$ such that $\left\langle 1\!\!1_{B'}, \mu^M \right\rangle < \eta$ for all $M \in \mathbb{N}$, i.e.

$$\lim_{\langle 1\!\!1_{B'}, \overline{\mu}\rangle \to 0} \left\langle 1\!\!1_{B'}, \mu^M \right\rangle \to 0 \text{ uniformly for all } M. \tag{3.10}$$

By Theorem IV.9.12 in [9], the sequence (μ^M) is weakly sequentially compact and there exists a subsequence of (μ^N) (again denoted by (μ^N)) such that (μ^N) converges weakly to some $\widetilde{\mu} \in \mathcal{M}_1^f$.
Step 6: First (3.7) shows that $\left\langle 1\!\!1_{\Omega\times[t,\infty)}, \widetilde{\mu} \right\rangle = 0$ and from (3.8) we conclude that for any $\eta > 0$ and any sequence $B_\kappa \in \mathcal{F}', B_\kappa \nearrow \Omega'$ for $\kappa \to \infty$ we have $\langle 1\!\!1_{B_\kappa}, \widetilde{\mu} \rangle \geq 1 - \eta$ for all sufficintly large κ. Therefore $\widetilde{\mu}$ is a probability measure absolutely continuous w.r.t. μ^0. Moreover, for every $\varepsilon' > 0$ there exists $Y \in \mathcal{C}'$ such that

$$\phi'^*(\widetilde{\mu}) \geq \langle Y, \widetilde{\mu} \rangle - \varepsilon' = \lim_{N\to\infty} \left\langle Y, \mu^N \right\rangle - \varepsilon' \geq \liminf_{N\to\infty} \phi'^*(\mu^N) - \varepsilon'.$$

From (3.6) we conclude that

$$\left\langle X, \mu^N \right\rangle - \phi'^*(\mu^N) < -\varepsilon$$

for all $N \in \mathbb{N}$ such that $\langle X, \widetilde{\mu} \rangle - \phi'^*(\widetilde{\mu}) < 0$. Transforming $\widetilde{\mu}$ back to $\widetilde{a} \in \mathcal{A}_1^1$ via the Radon-Nikodym density $\Delta\widetilde{a}_t := \frac{\partial\widetilde{\mu}(.\ \cap \Omega\times\{t\})}{\partial\mathbb{P}}$ restricted to $\Omega \times \{t\}$, we see that $\widetilde{a} \in \mathcal{A}_1^{1,loc}$ and

$$\langle X, \widetilde{a} \rangle - \phi^*(\widetilde{a}) < 0.$$

This shows that $\mathcal{C}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^{1,loc})$-closed.

$(ii) \Rightarrow (iii)$: The $\sigma\left(\mathcal{R}^\infty, \mathcal{A}^{1,\mathrm{loc}}\right)$-upper semicontinuity follows directly from (3.3) and (ii) of Definition 2.1.

$(iii) \Rightarrow (iv)$: This follows from the Fenchel-Moreau theorem.

$(iv) \Rightarrow (v)$: Fix $\varepsilon > 0$ and let (X^n) be a sequence in $\mathcal{R}^\infty$ and $X \in \mathcal{R}^\infty$ such that $\sup_{n\in\mathbb{N}} |X^n_t| \in L^\infty$ and $X^n_t \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$. There is $a^* \in \mathcal{A}^{1,\mathrm{loc}}_1$ such that

$$
\begin{aligned}
\phi(X) + \varepsilon &\geq \langle X, a^*\rangle - \phi^*(a^*) \\
&= \lim_{n\to\infty} \left(\langle X^n, a^*\rangle - \phi^*(a^*)\right) \\
&\geq \limsup_{n\to\infty} \inf_{a\in\mathcal{A}^{1,\mathrm{loc}}_1} \{\langle X^n, a\rangle - \phi^*(a)\} = \limsup_{n\to\infty} \phi(X^n).
\end{aligned}
$$

$(v) \Rightarrow (i)$: Let $X \in \mathcal{R}^\infty$ such that $X\mathbb{1}_{(0,t)} + N(t)\mathbb{1}_{[t,\infty)} \in \mathcal{C}$ for all $t \in \mathbb{N}$ and some large $N(t) \in \mathbb{N}$. The sequence (X^n) defined as

$$
X^n := X\mathbb{1}_{(0,n)} + N(n)\mathbb{1}_{[n,\infty)}
$$

satisfies $\phi(X^n) \geq 0$ for all $n \in \mathbb{N}$. Moreover, $\sup_{n\in\mathbb{N}} |X^n_t| \in L^\infty$ and $X^n_t \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$. Hence

$$
\phi(X) \geq \limsup_{n\to\infty} \phi(X^n) \geq 0
$$

which shows that $X \in \mathcal{C}$. □

Remark 3.4 Let ϕ be a risk assessment which is locally continuous from below and has the robust representation

$$
\phi(X) = \inf_{a\in\mathscr{A}} \{\langle X, a\rangle - \phi^*(a)\} \qquad \text{for all } X.
$$

The arguments in step 3 of the implication $(i) \Rightarrow (ii)$ show that for any $\gamma > 0$ the set

$$
\mathscr{A}^\gamma := \{a \in \mathscr{A} : \phi^*(a) \geq -\gamma\}
$$

is locally uniformly integrable in the following sense: For every $T \in \mathbb{N}$ and any sequence $B^n = (B^n_1, \ldots)$ satisfying $B^n_t \underset{n\to\infty}{\nearrow} \Omega$ for all $t \leq T$ and $B^n_t = \Omega$ for all $t > T$ one has

$$
\lim_{n\to\infty} \inf_{a\in\mathscr{A}^\gamma} \langle \mathbb{1}_{B^n}, a\rangle = 1
$$

where $\mathbb{1}_{B^n} = \left(\mathbb{1}_{B^n_1}, \ldots\right)$. Indeed, for any $\varepsilon > 0$ and $M \geq \gamma/\varepsilon$ we get

$$
\begin{aligned}
\lim_{n\to\infty} \inf_{a\in\mathscr{A}^\gamma} \langle \mathbb{1}_{B^n}, a\rangle &\geq \frac{1}{M} \lim_{n\to\infty} \inf_{a\in\mathcal{A}^\gamma} \left(\langle M \cdot \mathbb{1}_{B^n}, a\rangle - \phi^*(a)\right) - \varepsilon \\
&\geq \frac{1}{M} \lim_{n\to\infty} \phi\left(M \cdot \mathbb{1}_{B^n}\right) - \varepsilon = 1 - \varepsilon.
\end{aligned}
$$

Example 3.5 Let $\mathcal{P}$ be a uniformly integrable set of absolutely continuous probabilities and $\mathbb{T}$ a subset of $\mathbb{N}$. Then

$$\phi(X) := \inf_{t \in \mathbb{T}} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[X_t]$$

is an asymptotically stable coherent risk assessment which is locally continuous from below.

Let $\phi : \mathcal{R}^\infty \to \mathbb{R}$ be an asymptotically stable risk assessment which is locally continuous from below. Then it holds that

$$\phi(X) = \limsup_{t \to \infty} \phi(X 1\!\!1_{(0,t)} + X_t 1\!\!1_{[t,\infty)}). \tag{3.11}$$

However, the following example shows that there exist asymptotically stable risk assessments which are locally continuous from below, which satisfy

$$\phi(X) > \liminf_{t \to \infty} \phi(X 1\!\!1_{(0,t)} + X_t 1\!\!1_{[t,\infty)})$$

for some $X \in \mathcal{R}^\infty$, i.e. $\lim_{t \to \infty} \phi(X 1\!\!1_{(0,t)} + X_t 1\!\!1_{[t,\infty)})$ does not exist in general.

Example 3.6 Suppose $\Omega = \{\omega\}$ and $\phi(X) := \inf_{a \in \mathcal{Q}} \langle X, a \rangle$ where

$$\mathcal{Q} := \left\{ a \in \mathcal{A}_1^{1,loc} : \Delta a_t = \Delta a_{t+1} = \frac{1}{2} \text{ for some } t \in \mathbb{N} \right\}.$$

Then ϕ is an asymptotically stable coherent risk assessment which is locally continuous from below. However, for $X = (1, -1, 1, -1, 1, \dots)$ one has $\phi(X) = 0$, whereas $\liminf_{t \to \infty} \phi(X 1\!\!1_{(0,t)} + X_t 1\!\!1_{[t,\infty)}) = -1$.

4 Dynamic risk assessments

In this section, we study families of risk assessments $(\phi_s)_{s \in \mathbb{N}_0}$ which are time-consistent:

$$\phi_s(X) = \phi_s\left(X 1\!\!1_{(s,t]} + \phi_t(X) 1\!\!1_{(t,\infty)}\right), \quad \text{for all } s, t \in \mathbb{N} \text{ with } s < t. \tag{4.1}$$

The time-consistent property of a risk assessment is an important concept in multiperiod risk assessment (see [1], [2], [6]). In particular, within the problem of optimizing the assessment with respect to a control process, time-consistency implies the Bellman principle (see [2]). This principle allows to replace a global optimization problem over processes by a recursive procedure of timely local optimization problems over random variables. In the context of insurance risks together with a liquid market, one is interested in portfolios, which in addition to the insurance risk minimize the total risk assessment. Such portfolios are called optimal replicating portfolios (ORP). The time-consistency of the risk assessment reduces this problem to find an optimal portfolio just for one period (see [10] for concepts of insurance supervisions, like Solvency II or the Swiss Solvency Test). Furthermore, some accounting concepts for insurances, like the risk margin as external capital, are defined with respect to such optimal replicating portfolios.

Here we work with the notion of time-consistency as in [10] which is slightly different to the respective notion in [1] and [5]. Since both concepts of time-consistency are formally very similar, the results of this section can directly be adapted to the context of [1] and [5].

Time-consistent risk assessments lead naturally the the notion of a family of generators $G_s : L^\infty_{s+1} \times L^\infty_{s+1} \to L^\infty_s$ by defining

$$G_s(Z^1, Z^2) := \phi_s \left(Z^1 \mathbb{1}_{\{s+1\}} + Z^2 \mathbb{1}_{(s+1,\infty)} \right). \tag{4.2}$$

This gives

$$\phi_s(X) = G_s \left(X_{s+1}, \phi_{s+1}(X) \right). \tag{4.3}$$

Here the goal is to give conditions of a family $(G_s)_{s\in\mathbb{N}_0}$ of generators which leads to asymptotically stable risk assessments which are locally continuous from below. We start with the following properties of generators:

(G0) $G_s(0,0) = 0$,

(G1) $G_s(X + m, Y + m) = G_s(X, Y) + m$ for all $m \in L^\infty_s$,

(G2) $G_s(X^1, Y^1) \geq G_s(X^2, Y^2)$ whenever $X^1 \geq X^2$ and $Y^1 \geq Y^2$,

(G3) $G_s(X, Y) = \lim_{n\to\infty} G_s(X^n, Y^n)$ for every decreasing sequence (X^n, Y^n) which converges to some (X, Y) $\mathbb{P}$-almost surely,

(G3') $G_s(X, Y) = \lim_{n\to\infty} G_s(X^n, Y^n)$ for every increasing sequence (X^n, Y^n) which converges to some (X, Y) $\mathbb{P}$-almost surely,

(G4) $G_s(\lambda X^1 + (1-\lambda)X^2, \lambda Y^1 + (1-\lambda)Y^2) \geq \lambda G_s(X^1, Y^1) + (1-\lambda) G_s(X^2, Y^2)$ for all $\lambda \in L^\infty_s$ with $0 \leq \lambda \leq 1$.

Under the concavity assumption (G4) the condition (G3') implies (G3). For $X \in \mathcal{R}^\infty$, $s \in \mathbb{N}_0$ and $N \geq \|X\|_{\mathcal{R}^\infty}$ the sequence $\{G_s(X_{s+1}, \cdot) \circ \cdots \circ G_{t-1}(X_t, N)\}_{t\geq s+1}$ is nonincreasing in t and we define

$$\phi^N_s(X) := \inf_{t\geq s+1} G_s(X_{s+1}, \cdot) \circ \cdots \circ G_{t-1}(X_t, N).$$

Theorem 4.1 *Suppose that the generators $(G_s)_{s\in\mathbb{N}_0}$ satisfy (G0)–(G3) and*

$$\lim_{n\to\infty} G_s(0, \cdot) \circ \cdots \circ G_{s+n}(0, m) = 0 \quad \textit{for all } m \geq 0. \tag{4.4}$$

Then, for every $s \in \mathbb{N}_0$ one has

$$\phi_s(X) := \phi^N_s(X) = \phi^M_s(X) \quad \textit{for all } M, N \geq \|X\|_{\mathcal{R}^\infty},$$

and the family $(\phi_t)_{t\in\mathbb{N}_0}$ is time-consistent in the sense of (4.1). *Further, ϕ_0 satisfies (i)–(iii) of Definition 2.1, and $\phi_0(X) = \lim_{n\to\infty} \phi_0(X^n)$ for every sequence (X^n) in $\mathcal{R}^\infty$ and $X \in \mathcal{R}^\infty$ such that $X^n \geq X^{n+1}$ and $X^n_t \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$.*

Under the additional assumption (G4), ϕ_0 is a concave risk assessment. If the generators satisfy (G3') instead of (G3) then ϕ_0 is locally continuous from below.

Remark 4.2 For instance, condition (4.4) holds if for every $\varepsilon > 0$ there exists $\beta(\varepsilon)$ with $0 \le \beta(\varepsilon) < 1$ such that $G_s(0,m) \le \beta(\varepsilon)m$ for all $m \ge \varepsilon$ and $G_s(0,m) \le \varepsilon$ whenever $m < \varepsilon$ for eventually all s.

Proof: We first show that the definition of ϕ_s^N does not depend on $N \ge \|X\|_{\mathcal{R}^\infty}$. To that end, we fix $X \in \mathcal{R}^\infty$ and $M \ge N \ge \|X\|_{\mathcal{R}^\infty}$. For every $\varepsilon > 0$ there exists $t \in \mathbb{N}$ such that

$$\varepsilon + \phi_0^N(X) \ge G_0(X_1,\cdot) \circ \cdots \circ G_{t-1}(X_t, N).$$

In view of (G1) one has

$$G_t(N,\cdot) \circ \cdots \circ G_{t'-1}(N,M) = N + G_t(0,\cdot) \circ \cdots \circ G_{t'-1}(0, M-N)$$

for all $t' \ge t+1$. Thus, by condition (4.4), (G1) and (G2) there exists $t' \ge t$ such that

$$\begin{aligned}\phi_0^N(X) + 2\varepsilon &\ge G_0(X_1,\cdot) \circ \cdots \circ G_{t-1}(X_t, N) + \varepsilon\\ &\ge G_0(X_1,\cdot) \circ \cdots \circ G_{t-1}(X_t,\cdot) \circ G_t(N,\cdot) \circ \cdots \circ G_{t'-1}(N,M)\\ &\ge G_0(X_1,\cdot) \circ \cdots \circ G_{t-1}(X_t,\cdot) \circ G_t(X_{t+1},\cdot) \circ \cdots \circ G_{t'-1}(X_{t'},M)\\ &\ge \phi_0^M(X).\end{aligned}$$

Since $\phi_0^N(X) \le \phi_0^M(X)$ by (G2), we get $\phi_0^N(X) = \phi_0^M(X)$, which shows that ϕ_0 is well-defined. The argumentation for ϕ_s works analogously, however t and t' have to be chosen $\mathcal{F}_s$-measurable with values in $\mathbb{N}$.

As for the time-consistency (4.1), the conditions (G2) and (G3) imply

$$\begin{aligned}&\phi_s\left(X\mathbb{1}_{(s,t]} + \phi_t(X)\mathbb{1}_{(t,\infty)}\right)\\ =&G_s(X_{s+1},\cdot) \circ \cdots \circ G_{t-1}(X_t, \phi_t(X))\\ =&G_s(X_{s+1},\cdot) \circ \cdots \circ G_{t-1}(X_t,\cdot) \circ \left(\inf_{t' \ge t+1} G_t(X_{t+1},\cdot) \circ \cdots \circ G_{t'-1}(X_{t'},N)\right)\\ =&\inf_{t' \ge t+1} G_s(X_{s+1},\cdot) \circ \cdots \circ G_{t-1}(X_t,\cdot) \circ G_t(X_{t+1},\cdot) \circ \cdots \circ G_{t'-1}(X_{t'},N)\\ =&\phi_s(X),\end{aligned}$$

for all $N \ge \|X\|_{\mathcal{R}^\infty}$.

To show that ϕ_0 is continuous from above, let (X^n) be a decreasing sequence in $\mathcal{R}^\infty$ such that $N := \sup_{n\in\mathbb{N}} \|X^n\|_{\mathcal{R}^\infty} < \infty$ and $X_t^n \to X_t$ P-almost surely for all $t \in \mathbb{N}$ for some $X \in \mathcal{R}^\infty$. For every $\varepsilon > 0$ there exists a time horizon $t \in \mathbb{N}$ such that

$$\begin{aligned}\phi_0(X) + \varepsilon &\ge G_0(X_1,\cdot) \circ \cdots \circ G_{t-1}(X_t,N))\\ &= \lim_{n\to\infty} G_0(X_1^n,\cdot) \circ \cdots \circ G_{t-1}(X_t^n,N)\\ &\ge \lim_{n\to\infty} \phi_0(X^n).\end{aligned}$$

On the other hand, $\phi_0(X^n) \ge \phi_0(X)$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} \phi_0(X^n) = \phi_0(X)$.

In case that the generators satisfy (G3'), for all $T \in \mathbb{N}$ we have

$$\begin{aligned}\phi_0(X\mathbb{1}_{(0,T)} + X_T\mathbb{1}_{[T,\infty)}) &= G_0(X_1,\cdot)\circ\cdots\circ G_{T-1}(X_T,X_T)) \\ &= \lim_{n\to\infty} G_0(X_1^n,\cdot)\circ\cdots\circ G_{T-1}(X_T^n,X_T^n) \\ &= \lim_{n\to\infty} \phi_0(X^n\mathbb{1}_{(0,T)} + X_T^n\mathbb{1}_{[T,\infty)})\end{aligned}$$

for every sequence (X^n) in $\mathcal{R}_T^\infty$ which increases to some $X \in \mathcal{R}_T^\infty$, which shows that ϕ_0 is locally continuous from below. □

Proposition 4.3 *Suppose that the generators $(G_s)_{s\in\mathbb{N}_0}$ satisfy (G0)–(G3') and condition* (4.4)*. If in addition*

$$\limsup_{t\to\infty} \sup_{N\in\mathbb{N}} G_t(0,N) < \infty, \tag{4.5}$$

then ϕ_0 is asymptotically stable.

Proof: For every $X \in \mathcal{R}^\infty$ and $N \in \mathbb{N}$ one has

$$\begin{aligned}\phi_0\left(X\mathbb{1}_{(0,t]} + N\mathbb{1}_{(t,\infty)}\right) &= \phi_0\left(X\mathbb{1}_{(0,t)} + \phi_{t-1}\left(X\mathbb{1}_{\{t\}} + N\mathbb{1}_{(t,\infty)}\right)\mathbb{1}_{[t,\infty)}\right) \\ &\le \phi_0\left(X\mathbb{1}_{(0,t)} + G_{t-1}(\|X\|_{\mathcal{R}^\infty}, \|X\|_{\mathcal{R}^\infty} + N)\mathbb{1}_{[t,\infty)}\right) \\ &\le \phi_0\left(X\mathbb{1}_{(0,t)} + [\|X\|_{\mathcal{R}^\infty} + G_{t-1}(0,N)]\,\mathbb{1}_{[t,\infty)}\right)\end{aligned}$$

for all $t \in \mathbb{N}$. Hence, by (4.5) there exist $t_0 \in \mathbb{N}$ and a constant $C \in \mathbb{N}$ such that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left(\phi_0\left(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}\right) - \phi_0(X)\right) \\ &\quad\le \phi_0\left(X\mathbb{1}_{(0,t)} + (\|X\|_{\mathcal{R}^\infty} + C)\,\mathbb{1}_{[t,\infty)}\right) - \phi_0(X)\end{aligned}$$

for all $t \ge t_0$. Since ϕ_0 is continuous from above, the right hand side tends to zero as t goes to infinity. Hence

$$\lim_{t\to\infty}\sup_{N\in\mathbb{N}}\left(\phi_0\left(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}\right) - \phi_0(X)\right) = 0,$$

showing that ϕ_0 satisfies (A2). □

5 Examples

In this section, we construct examples of generators which satisfy the conditions (G0)–(G3'), (4.4) and (4.5). To that end, we consider generators of the form

$$G_s(X,Y) := \psi_s\left(X + h_s(Y-X)\right), \quad s \in \mathbb{N}_0, \tag{5.1}$$

where $\psi_s : L_{s+1}^\infty \to L_s^\infty$ such that

(p0) $\psi_s(0) = 0$,

(p1) $\psi_s(Z+m) = \psi_s(Z) + m$ for all $m \in L_s^\infty$,

(p2) $\psi_s(Z^1) \geq \psi_s(Z^2)$ whenever $Z^1 \geq Z^2$,

(p3) $\psi_s(Z^n) \to \psi_s(Z)$ for every sequence (Z^n) which increases to Z,

(p4) $\psi_s(\lambda Z^1 + (1-\lambda)Z^2) \geq \lambda\psi_s(Z^1) + (1-\lambda)\psi_s(Z^2)$ for all $\lambda \in L_s^\infty$ with $0 \leq \lambda \leq 1$,

and the function $h_s : \mathbb{R} \to \mathbb{R}$ satisfies

(h0) $h_s(0) = 0$,

(h1) $h_s(z+m) \leq h_s(x) + m$ for all $z \in \mathbb{R}$ and $m \geq 0$,

(h2) $h_s(z^1) \geq h_s(z^2)$ whenever $z^1 \geq z^2$,

(h3) h_s is continuous,

(h4) h_s is concave.

A straightforward application of Theorem 4.1, Remark 4.2 and Proposition 4.3 is then:

Proposition 5.1 *Let $(G_s)_{s\in\mathbb{N}_0}$ be a sequence of generators of the form* (5.1) *which satisfy (p0)–(p4), (h0)–(h4), for every $\varepsilon > 0$ there exists $0 \leq \beta(\varepsilon) < 1$ such that $h_s(m) \leq \beta(\varepsilon)m$ for all $m \geq \varepsilon$ and $h_s(m) \leq \varepsilon$ whenever $m < \varepsilon$, and*

$$\limsup_{s\to\infty} \sup_{z\in\mathbb{N}} h_s(z) < \infty.$$

Then the generators G_s satisfy (G0)–(G2), (G3'), (G4). The corresponding concave risk assessment ϕ_0 is locally continuous from below and asymptotically stable.

Proof: Clearly, such generators G_s satisfy (G0) and (G1). As for the monotonicity (G2), for $X^1 \geq X^2$ and $Y^1 \geq Y^2$ we have $X^2 + h_s(Y^2 - X^2) \leq X^1 + h_s(Y^2 - X^1)$ by (h1) so that

$$\begin{aligned} G_s(X^1, Y^1) &= \psi_s(X^1 + h_s(Y^1 - X^1)) \geq \psi_s(X^1 + h_s(Y^2 - X^1)) \\ &\geq \psi_s(X^2 + h_s(Y^2 - X^2)) = G_s(X^2, Y^2). \end{aligned}$$

To show (G3') let (X^n, Y^n) be a sequence which increases to (X, Y). Then $X^n + h_s(Y^n - X^n)$ increases to $X + h_s(Y - X)$ by (h1)–(h3) so that $G_s(X^n, Y^n)$ increases to $G_s(X, Y)$. By (p2) and (h4), it holds that

$$\begin{aligned} &\psi_s\left(\lambda X^1 + (1-\lambda)X^2 + h_s(\lambda(Y^1 - X^1) + (1-\lambda)(Y^2 - X^2))\right) \\ &\geq \psi_s\left(\lambda X^1 + (1-\lambda)X^2 + \lambda h_s(Y^1 - X^1) + (1-\lambda)h_s(Y^2 - X^2)\right) \\ &\geq \lambda\psi_s(X^1 + h_s\left(Y^1 - X^1\right)) + (1-\lambda)\psi_s(X^2 + h_s\left(Y^2 - X^2\right)), \end{aligned}$$

which shows (G4).

Finally, that ϕ_0 is an asymptotically stable risk assessment follows from Theorem 4.1 and Proposition 4.3 since

$$G_s(0,m) = \psi_s(h_s(m)) = h_s(m)$$

implies (4.4) by Remark 4.2 and (4.5). □

Notice that Proposition 5.1 allows to construct time-consistent risk assessments which are asymptotically stable. For instance, the generators

$$G_s(X,Y) = \psi_s\left(X + h_s(Y-X)\right)$$

can be defined through the negative of a conditional risk measure ψ_s such as the entropic utility function

$$\psi_s(Z) = \frac{1}{\gamma}\log\left(\mathbb{E}\left[\exp(-\gamma Z) \mid \mathcal{F}_s\right]\right)$$

with risk aversion parameter γ, and discounting functions such as

- $h_s(z) = -z^-$
- $h_s(z) = 1 - \exp(-z^+) - z^-$,

where $z^+ := \max(x,0)$ and $z^- := \max(-z,0)$, and which both satisfy the assumptions of Proposition 5.1. Alternatively, ψ_s could be chosen as the negative of the conditional Value at Risk for which (p4) does not hold, or the conditional Average Value at Risk.

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