Approaches to conditional risk*

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Abstract

We present and compare two different approaches to conditional risk measures. One approach draws from convex analysis in vector spaces and presents risk measures as functions on L^p spaces, while the other approach utilizes module-based convex analysis where conditional risk measures are defined on L^p type modules. Both approaches utilize general duality theory for vector–valued convex functions in contrast to the current literature in which we find ad hoc dual representations. By presenting several applications such as monotone and (sub)cash invariant hulls with corresponding examples we illustrate that module-based convex analysis is well suited to the concept of conditional risk measures.

Keywords: Conditional risk measures, L^0–modules, L^p–type modules, Monotone hulls, Subcash invariant hulls, Cash invariant hulls

JEL Classification: C60

1 Introduction

When [ADEH99] introduced the notion of monetary risk measures they inspired a lively and fruitful discussion about duality theory of risk measures in financial mathematics, cf. [CL08, Del00, Del02, Del06, FS08b, FS02, FS04, FRG02, KR09, RS06, Web06] and the various references therein. Subsequently, there have been many contributions addressing the question how dual representation

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results for real-valued risk measures translate into conditional and eventually dynamic discrete time frameworks, cf. [ADE$^+$07, BN04, CDK06, DS05, FP06, Rie04].

Within these articles, a technique referred to as scalarization is commonly applied to establish dual representation results for conditional risk measures in an ad hoc manner. The corresponding proofs are performed by reducing the originally given conditional problem to the static case in a first step. In a second step one applies standard duality theory, and in the third step one translates the results obtained back into the multi period framework. As a consequence, many intuitive structures are disguised.

The aim of the present article is to present two different approaches to duality theory of conditional risk measures which do not follow the ad hoc path. In contrast to the literature, both approaches start and remain on the "conditional level" by utilizing duality theory for vector valued functions. Thereby, the scalarization method is avoided for the convex analysis, but is used instead for a more fundamental representation result of some special linear operators on L^p spaces (Proposition 2.5). The results become more natural and their proofs are intuitive.

The two approaches differ in one fundamental way. The first one is vector space based and therefore closer to what has been established in the literature so far. Within the second one, vector space theory is only of minor interest as this approach assumes modules as the naturally underlying structure in a framework with contingent initial data. Both approaches reveal the key properties of conditional risk measures in contrast to general convex functions. Especially the module approach leads to a theory almost entirely analogous to that of static risk measures.

The present article is conceptual in nature with a focus on intuition. The ideas of most of the proofs will seem familiar to anyone who is familiar with the theory of static risk measures. Nevertheless, it requires non trivial machinery from vector and module based duality theory. This article shall be seen as an application of the theory established in [FKV09] and [KV09] to conditional risk measures. In fact, it provides a financial motivation for the module based convex analysis presented in [FKV09] and [KV09].

The remainder of this paper is as follows. In Section 2 we introduce conditional risk measures on L^p spaces. This approach is vector space based and it extends the current literature where conditional risk measures are studied on the significantly smaller Banach space L^∞. This approach draws from a general vector space duality result, established in [Zow75]. As outlined above, this result forms the base of our observations from which we will subsequently derive more specific results for conditional risk measures. This approach can be regarded as a top down approach as it originates from a dual representation result for general vector valued convex functions and then reveals how additional properties of the represented functions translate to properties of the representing continuous linear functions. This translation is of particular interest in the context of conditional risk measures as it clarifies under which conditions the represented convex function can be interpreted as the maximum of expected losses under different scenarios possibly subject to penalization.

In Section 3 we present a module based approach to duality theory of conditional risk measures. In contrast to Section 2 the spirit of this approach can be referred to as bottom up. The reason for this is that from the beginning on we establish that continuous module homomorphisms, which now take the place of continuous linear functions, are necessarily conditional expectations. As a consequence, dual representations of conditional risk measures can immediately be interpreted as the maximum of expected losses subject to penalization. It is due to this approach that the discussion of Section 2 becomes obsolete to a large extent. Nevertheless, this comes at the cost of module based convex analysis which is a technically involved matter. The main advantage of this approach however is that the derived duality theory for conditional risk measures is very similar to that of static risk measures.

In Section 4 we present further applications of module based duality theory to conditional risk measures and thereby illustrate further advantages of the module approach over the vector space one. The aim of this section is to approximate convex functions by monotone and (sub)cash invariant functions. Duality theory is utilized to find a monotone and (sub)cash invariant function "closest", expressed in dual terms, to a given function. These approximating functions are called monotone and (sub)cash invariant hulls. The idea of this duality based construction principle is already presented in [FK07] which, however, only covers the static case.

In Section 5 we present examples of convex functions and their monotone (sub)cash invariant hulls and explicitly construct their subgradients. The purpose of this section is to illustrate the theory.

Throughout this article, we fix a probability space $(\Omega, \mathcal{E}, P)$ as stochastic basis. By $L^0(\mathcal{G})$ we denote the space of real valued $\mathcal{G}$–measurable random variables, where $\mathcal{G} \subset \mathcal{E}$ is a generic sub σ–algebra, and we note that $L^0(\mathcal{G})$ is also a ring. Random variables and measurable sets which coincide almost surely are identified. Equalities and inequalities between random variables are understood in the almost sure sense. Further, $L^0_+(\mathcal{G}) = \{X \in L^0(\mathcal{G}) \mid X \geq 0\}$, $L^0_{++}(\mathcal{G}) = \{X \in L^0(\mathcal{G}) \mid X > 0\}$. $\bar{L}^0(\mathcal{G})$ denotes the set of $\mathcal{G}$–measurable random variables which take values in $\mathbb{R} \cup \{\pm\infty\}$ and $\bar{L}^0_+(\mathcal{G}) = \{X \in \bar{L}^0(\mathcal{G}) \mid X \geq 0\}$. Further, we consider non trivial initial information given by a σ–algebra $\mathcal{F} \subset \mathcal{E}$. Throughout, we define $0 \cdot (\pm\infty) = 0$.

2 The vector space approach

For all of this section we fix $1 \leq r \leq p < \infty$. We denote by s and q the respective dual exponents of r and p. That is, $s = r/(r-1)$, $q = p/(p-1)$ with the convention $s, q = \infty$ if $r, p = 1$. By $L^k(\mathcal{G}) = L^k(\Omega, \mathcal{G}, P)$ we denote the space of $\mathcal{G}$–measurable functions with finite kth moments, that is,

$$L^k(\mathcal{G}) = \{X \in L^0(\mathcal{G}) \mid E[|X|^k] < +\infty\}$$

where $\mathcal{G} \subset \mathcal{E}$ denotes a generic sub σ–algebra of $\mathcal{E}$ and $k \in [1, +\infty)$. $L^\infty(\mathcal{G}) = L^\infty(\Omega, \mathcal{G}, P)$ denotes the space of essentially bounded $\mathcal{G}$–measurable random variables.

In this paper we do not cover the case of $p = +\infty$. The reason for this is that numerous articles from the vast literature on financial risk measures deal with conditional risk measures on $L^\infty(\mathcal{E})$; we refer to [ADE⁺07, BN04, CDK06, DS05, FP06, Rie04] and the references therein.

2.1 Preliminaries

Definition 2.1. *A function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *monotone if $f(X) \leq f(X')$ for all $X, X' \in L^p(\mathcal{E})$ with $X \geq X'$,*

(ii) *subcash invariant if $f(X + Y) \geq f(X) - Y$ for all $X \in L^p(\mathcal{E})$ and $Y \in L^\infty(\mathcal{F})$ with $Y \geq 0$,*

(iii) *cash invariant if $f(X+Y) = f(X)-Y$ for all $X \in L^p(\mathcal{E})$ and $Y \in L^\infty(\mathcal{F})$.*

Recall that a function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is convex if $f(\alpha X + (1-\alpha)X') \leq \alpha f(X) + (1-\alpha)f(X')$ for all $X, X' \in L^p(\mathcal{E})$ and $\alpha \in [0,1]$. f is local if

$$1_A f(X) = 1_A f(1_A X) \text{ for all } X \in L^p(\mathcal{E}) \text{ and } A \in \mathcal{F}. \tag{1}$$

In line with the literature, we refer to a convex function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ which is monotone and cash invariant as a conditional (monetary) risk measure. The reason for this is the economic interpretation of $f(X)$ as a capital requirement a financial institution has to meet on assuming the uncertain profit $X \in L^p(\mathcal{E})$ adherent to a financial position.

By the Riesz representation theorem any continuous linear function $\mu : L^p(\mathcal{E}) \to \mathbb{R}$ is of the form

$$\mu X = E[ZX]$$

for some $Z \in L^q(\mathcal{E})$. Further, any proper lower semicontinuous (l.s.c.) convex function $f : L^p(\mathcal{E}) \to (-\infty, +\infty]$ admits the Fenchel–Moreau dual representation

$$f(X) = \sup_{Z \in L^q(\mathcal{E})} (E[ZX] - f^*(Z)), \tag{2}$$

where $f^*(Z) = \sup_{X \in L^p(\mathcal{E})}(E[ZX] - f(X))$ denotes the conjugate function of f.

Dual representations as in (2) and subdifferentiability are of distinct interest in various contexts such as optimal investment problems with respect to robust utility functionals [SW05, Sch07], portfolio optimization under risk constraints [GW07, GW08], risk sharing [BEK05, BR06, Acc07, BR08, FS08a, FK08, JST08, LR08, Che09], equilibrium pricing [KS07, FK08], efficient hedging [FL00, Rud07, Che09, İJS09] as well as the numerous references therein.

Moreover, such representations provide us with a plausible interpretation of the subjective risk assessment of an economic agent. More precisely, let us assume an agent faces the uncertain payoff $X \in L^p(\mathcal{E})$. Dual representations of the form (2) suggest that the agent computes the expected payoff $E[ZX]$ within the specific model $Z \in L^q(\mathcal{E})$ selected from a variety of probabilistic models which are penalized by $-f^*(Z)$. The higher $f^*(Z)$ the less plausible the agent views model Z. In evaluating the capital requirement $f(X)$ for the uncertain payoff X the agent then takes a worst case approach.

For these reasons, the question arises as to what extent representations of the form (2) are preserved in the context of conditional risk measures when $\mathbb{R}$ is replaced by $L^r(\mathcal{F})$.

To address this question, we denote by $\mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$ the space of all continuous linear functions from $L^p(\mathcal{E})$ into $L^r(\mathcal{F})$ and consider a function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$. We define $f^* : \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F})) \to \bar{L}^0(\mathcal{F})$ by

$$f^*(\mu) = \operatorname*{ess.sup}_{X \in L^p(\mathcal{E})} (\mu X - f(X))$$

and $\mathrm{dom} f^* = \{\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F})) \mid f^*(\mu) \in L^r(\mathcal{F})\}$. By convention, the essential supremum of an empty family of random variables is $-\infty$. Further, we define $f^{**} : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ by

$$f^{**}(X) = \operatorname*{ess.sup}_{\mu \in \mathrm{dom} f^*} (\mu X - f^*(\mu)).$$

An element $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$ is a subgradient of f at $X_0 \in L^p(\mathcal{E})$ if

$$\mu(X - X_0) \leq f(X) - f(X_0)$$

for all $X \in L^p(\mathcal{E})$.

The set of all subgradients of f at X_0 is called the subdifferential of f at X_0 and denoted by $\partial f(X_0)$. By definition of the subdifferential $\partial f(X_0)$ we have the well known relation

$$\mu_0 \in \partial f(X_0) \text{ if and only if } \mu_0 \in \mathrm{dom} f^* \text{ and } f(X_0) = \mu_0 X_0 - f^*(\mu_0). \quad (3)$$

It should be noted that in Section 3.1 below we encounter slightly different notion of conjugate functions, effective domains and subdifferentials. Nevertheless, there will be no source of ambiguity as the respective sections are entirely self contained.

Example 2.2. *Let us assume that $\mathcal{F} = \sigma(A_n)$ is generated by a countable partition (A_n) of Ω (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{n \in \mathbb{N}} A_n = \Omega$). In this case, we can identify $L^r(\mathcal{F})$ with $l^r(\mathcal{F})$, the space of all real valued sequences (x_n) with $\sum_{n=1}^{\infty} p_n |x_n|^r < \infty$, where $p_n = P[A_n]$ for all $n \geq 1$. Hence, any function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is of the form*

$$f = (f_1, f_2, f_3, \ldots)$$

with a sequence of functions $f_n : L^p(\mathcal{E}) \to \mathbb{R}$, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} p_n |f_n(X)|^r < \infty$ for all $X \in L^p(\mathcal{E})$.

Localness of the function f is now reflected by the intuitive relation

$$1_{A_n} f(X) = (\underbrace{0, \ldots, 0}_{n-1\text{–times}}, f_n(X), 0, \ldots) = (\underbrace{0, \ldots, 0}_{n-1\text{–times}}, f_n(1_{A_n} X), 0, \ldots) \text{ for all } n \in \mathbb{N},$$

that is, the nth component f_n of f only depends on the coordinate spanned by the vector 1_{A_n}.

Example 2.3. *The local structure of Example 2.2 becomes even more apparent if $\mathcal{E}$ is generated by a finite partition $B_1, \ldots, B_n$ of Ω. In this case, $A_j = \bigcup_{i \in I_j} B_i$, where $\{1, \ldots, n\} = \bigcup_{1 \le j \le m} I_j$ so that $L^p(\mathcal{E}) = L^0(\mathcal{E}) = \mathbb{R}^n$ as well as $L^r(\mathcal{F}) = L^0(\mathcal{F}) = \mathbb{R}^m$.*

The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is now of the form $f = (f_1, \ldots, f_m)$ with arbitrary functions $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$. Localness of f now means that for each $1 \le j \le m$ the function f_j only depends on the coordinates I_j. We abuse notation and identify f_j with its restriction to the coordinates I_j. In other words, $f = (f_1, \ldots, f_m)$ for functions $f_1 : \mathbb{R}^{I_1}, \ldots, f_m : \mathbb{R}^{I_m} \to \mathbb{R}$ (after rearranging the coordinates $1, \ldots, n$ suitably).

Moreover, if f is $C^1(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\mathrm{D}f(X) = \begin{pmatrix} Df_1(X_{I_1}) & 0 & \cdots & 0 \\ 0 & Df_2(X_{I_2}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Df_m(X_{I_m}) \end{pmatrix}$$

for all $X \in \mathbb{R}^n$. (Note that the zeroes in the above matrices are understood as generic vector zeroes possibly differing in their dimensions.)

Zowe proves in [Zow75] the following dual representation result which, in fact, he establishes in a more general setup.

Theorem 2.4. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a convex function. If f is continuous at $X_0 \in L^p(\mathcal{E})$ then $\partial f(X_0) \neq \emptyset$ and*

$$f(X_0) = f^{**}(X_0). \tag{4}$$

For the sake of completeness, we provide a self contained proof in the Appendix A, tailored to our setup.

The relevant questions can now be specified as follows. Which linear $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is of the form

$$\mu X = E[ZX \mid \mathcal{F}] \tag{5}$$

for some $Z \in L^q(\mathcal{E})$? And further, for which convex $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is each $\mu \in \mathrm{dom} f^*$ of the form (5) so that

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} \left(E[ZX \mid \mathcal{F}] - f^*(Z) \right), \tag{6}$$

where $f^*(Z)$ is understood as $f^*(E[Z\cdot \mid \mathcal{F}])$?

2.2 Linear functions on $L^p(\mathcal{E})$

In this section we study representation results and corresponding continuity properties of linear functions from $L^p(\mathcal{E})$ to $L^r(\mathcal{F})$. The results are of preliminary nature for the following section on convex functions.

Proposition 2.5. *A function $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *continuous linear and*

(ii) *local*

if and only if it is of the the form (5) *for some unique representing $Z \in L^q(\mathcal{E})$ which satisfies the integrability condition $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$, where $r(p-1)/(p-r)$ is understood as $+\infty$ if $p = r$.*

Proof. To prove the if statement, let $\mu = E[Z\cdot \mid \mathcal{F}]$, $Z \in L^q(\mathcal{E})$ with $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{(p-r)}}(\mathcal{F})$. Inspection shows that μ is linear and local. To establish continuity we assume $1 < r < p$, the other cases work analogously. By Hölder's inequality

$$\begin{aligned} E\left[|E[ZX \mid \mathcal{F}]|^r\right] &\leq E\left[E[|Z|^q \mid \mathcal{F}]^{\frac{r}{q}} E[|X|^p \mid \mathcal{F}]^{\frac{r}{p}}\right] \\ &\leq E\left[E[|Z|^q \mid \mathcal{F}]^{\frac{pr}{q(p-r)}}\right]^{\frac{p-r}{p}} E\left[|X|^p\right]^{\frac{r}{p}}. \end{aligned}$$

Since

$$E\left[|Z|^q \mid \mathcal{F}\right]^{\frac{pr}{q(p-r)}} = E\left[|Z|^q \mid \mathcal{F}\right]^{\frac{r(p-1)}{p-r}} \in L^1(\mathcal{F}),$$

we deduce $\|E[ZX \mid \mathcal{F}]\|_r \leq c\|Z\|_p$ for some $c \in \mathbb{R}_+$. Hence, μ is continuous.

Conversely, if $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is a continuous linear function then so is $E \circ \mu : L^p(\mathcal{E}) \to \mathbb{R}$ and by the Riesz representation theorem there is $Z \in L^q(\mathcal{E})$ such that $E[\mu X] = E[ZX]$ for all $X \in L^p(\mathcal{E})$. Since μ is local we derive $E[1_A \mu X] = E[\mu(1_A X)] = E[Z 1_A X]$ for all $A \in \mathcal{F}$ and $X \in L^p(\mathcal{E})$. Thus, $\mu X = E[ZX \mid \mathcal{F}]$ for all $X \in L^p(\mathcal{E})$. It remains to show that $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{(p-r)}}(\mathcal{F})$. We distinguish between two different cases. If $r = 1$ then $E[|Z|^q \mid \mathcal{F}] \in L^1(\mathcal{F})$ as $E[|E[|Z|^q \mid \mathcal{F}]|] = E[|Z|^q] \in \mathbb{R}$. It remains to show the case $1 < r \leq p$. To this end, consider the adjoint $\mu' : L^s(\mathcal{F}) \to L^q(\mathcal{E})$ of μ. By definition,

$$(\mu' Y)(X) = E[Y E[ZX \mid \mathcal{F}]] = E[YZX], \quad X \in L^p(\mathcal{E}). \tag{7}$$

Since $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is continuous so is $\mu' : L^s(\mathcal{F}) \to L^q(\mathcal{E})$, and consequently, for all $Y \in L^s(\mathcal{F})$

$$\|\mu' Y\|_q \leq c\|Y\|_s,$$

for some real constant c. Since the L^q–norm coincides with the corresponding operator norm we find that for all $Y \in L^s(\mathcal{F})$

$$\sup_{X \in L^p(\mathcal{E}), \|X\|_p \leq 1} |(\mu' Y)(X)| = \sup_{X \in L^p(\mathcal{E}), \|X\|_p \leq 1} E[YZX] \leq c E[|Y|^s]^{\frac{1}{s}}. \tag{8}$$

With equation (7) we know that $E[YZ\cdot]$ is a continuous linear function from $L^p(\mathcal{E})$ to $\mathbb{R}$. Since the topological dual of $L^p(\mathcal{E})$ can be identified with $L^q(\mathcal{E})$ we see that necessarily $YZ \in L^q(\mathcal{E})$. Therefore, we can define

$$X_Y = \operatorname{sign}(YZ) \times |YZ|^{\frac{1}{(p-1)}} / E[|YZ|^q]^{\frac{1}{p}}$$

(with the convention $0/0 = 0$). Then $X_Y \in L^p(\mathcal{E})$ and $\|X_Y\|_p \leq 1$ for all $Y \in L^s(\mathcal{F})$. Hence, we conclude from (8) that for all $Y \in L^s(\mathcal{F})$

$$E[|YZ|^q]^{\frac{1}{q}} = E[YZX_Y] \leq cE[|Y|^s]^{\frac{1}{s}}.$$

In particular, $Y \mapsto E[E[|Z|^q \mid \mathcal{F}]Y]$ is a linear, continuous function from $L^{\frac{s}{q}}(\mathcal{F})$ to $\mathbb{R}$. Again, since the topological dual of $L^{\frac{s}{q}}(\mathcal{F})$ can be identified with $L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$ we find that necessarily $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$. $\square$

The next proposition provides a different set of conditions that are sufficient for μ to be of the form (5). These conditions spotlight the emphasis on conditional risk measures.

Proposition 2.6. *A function* $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$

(i) *is continuous linear,*

(ii) *satisfies* $\mu Y \geq -Y$ *for all* $Y \in L^\infty(\mathcal{F})$ *with* $Y \geq 0$*, and*

(iii) *and is monotone, i.e.* $\mu X \leq 0$ *for all* $X \in L^p(\mathcal{E})$*,* $X \geq 0$*,*

if and only if it is of the form (5) *for some representing* $Z \in L^q(\mathcal{E})$ *with* $E[Z \mid \mathcal{F}] \geq -1$ *and* $Z \leq 0$ *and which satisfies* $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{(p-r)}}(\mathcal{F})$*.*

Proof. The if statement follows by inspection, where continuity follows as in Proposition 2.5.

As to the only if statement we show that (i), (ii) and (iii) imply that μ is local. To this end, let $X \in L^p(\mathcal{E})$ be essentially bounded in a first step. Then $X \leq 1_A X + \|X - 1_A X\|_\infty$, where for $X' \in L^p(\mathcal{E})$,

$$\|X'\|_\infty = \operatorname{ess.inf}\{Y \in L^0(\mathcal{F}) \mid Y \geq |X'|\}.$$

Since μ is positive and $\mu Y \geq -Y$ for all $Y \in L^\infty(\mathcal{F})$ with $Y \geq 0$ we derive

$$\mu X \geq \mu(1_A X + \|X - 1_A X\|_\infty) \geq \mu(1_A X) - \|X - 1_A X\|_\infty.$$

On exchanging X and $1_A X$ we derive

$$|1_A \mu X - 1_A \mu(1_A X)| = 1_A |\mu X - \mu(1_A X)| \leq 1_A \|X - 1_A X\|_\infty = 0.$$

Thus, μ is local for all essentially bounded X. By a standard truncation and approximation argument we conclude that μ is local for all $X \in L^p(\mathcal{E})$. Thus, we established that μ is continuous linear local and hence by Proposition 2.5 of the form (5) for some representing $Z \in L^q(\mathcal{E})$ which satisfies the desired integrability condition. Further, (ii) and (iii) imply $E[Z \mid \mathcal{F}] \geq -1$ and $Z \leq 0$. $\square$

Remark 2.7. *Proposition 2.6 remains valid if* (ii) is replaced by the projection property

$$\mu Y = -Y \text{ for all } Y \in L^\infty(\mathcal{F}),$$

and $E[Z \mid \mathcal{F}] \geq -1$ is replaced by $E[Z \mid \mathcal{F}] = -1$.

Example 2.8. *Property* (iii) *is needed in Proposition 2.6, as the following example shows. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{E} = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\})$, $P[\omega_1] = 1/2$, $P[\omega_2] = P[\omega_3] = 1/4$ and $\mathcal{F} = \sigma(A_1, A_2)$ with $A_1 = \{\omega_1\}$ and $A_2 = \{\omega_2, \omega_3\}$. Define the random variables*

$$Z_1 = (-2, 1, -1), \quad Z_2 = (0, -2, -2)$$

and the linear map $\mu : L^0(\mathcal{E}) \to L^0(\mathcal{F})$ by

$$\mu(X) = \sum_{i=1}^{2} E[Z_i X] 1_{A_i}.$$

Then μ satisfies (i) *and* (ii) *of Proposition 2.6, but not* (iii) *since $\mu(0, 4, 0) = (1, -2, -2)$.*

Now suppose μ were of the form (5) *for some (not necessarily positive) $Z \in L^0(\mathcal{E})$. This implies, in particular, that*

$$E[1_{A_1} \mu X] = E[1_{A_1} Z X]$$

for all $X \in L^0(\mathcal{E})$. But for $X = (0, 4, 0)$ we obtain zero on the right hand side and $1/2$ on the left hand side, which is absurd. Hence μ cannot be of the form (5)*.*

2.3 Monotone (sub)cash invariant convex functions on $L^p(\mathcal{E})$

Given the results of the preceding section we now turn our attention to convex functions.

Lemma 2.9. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a function.*

(i) *If f is local then every $\mu \in \mathrm{dom} f^*$ is local.*

(ii) *If f is monotone then μ is monotone for each $\mu \in \mathrm{dom} f^*$.*

(iii) *If f is subcash invariant then $\mu Y \geq -Y$ for all $Y \in L^\infty(\mathcal{F})$ with $Y \geq 0$ for each $\mu \in \mathrm{dom} f^*$.*

(iv) *If f is cash invariant then $-\mu$ is a projection for each $\mu \in \mathrm{dom} f^*$.*

Proof. (i) Take a non local $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$. Then there are $X \in L^p(\mathcal{E})$, $A, B \in \mathcal{F}$, $A \subset B$, with $P[A] > 0$ and $\mu(1_B X) < \mu X$ on A. Then $\mu(-1_{B^c} X) = \mu(1_B X - X) < 0$ on A or, equivalently, $\mu(1_{B^c} X) > 0$ on A. This implies for all $n \in \mathbb{N}$

$$\mu(1_{B^c} nX) - f(1_{B^c} nX) = n\mu(1_{B^c} X) - f(0)$$

on A. As n tends to ∞, we conclude $\mu \notin \mathrm{dom} f^*$.

(ii) Let $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$ and suppose there is $X \geq 0$ such that $\mu X > 0$ with positive probability. By monotonicity of f, $f(nX) \leq f(X)$ for all $n \in \mathbb{N}$. Hence,

$$f^*(\mu) \geq \mu(nX) - f(nX) \geq n\mu X - f(X)$$

for all $n \in \mathbb{N}$. This implies $\mu \notin \mathrm{dom} f^*$.

(iii) Let $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$. By subcash invariance of f we have

$$\begin{aligned}
f^*(\mu) \quad &\geq \quad \operatorname*{ess.sup}_{X \in L^p(\mathcal{E})} (\mu(X) - f(X + nY) - nY) \\
&\overset{X'=X+nY}{=} \quad \operatorname*{ess.sup}_{X' \in L^p(\mathcal{E})} (\mu(X' - nY) - f(X') - nY) \\
&= \quad \operatorname*{ess.sup}_{X' \in L^p(\mathcal{E})} (\mu(X') - f(X') + n(-\mu Y - Y)) \\
&= \quad f^*(\mu) + n(-\mu Y - Y)
\end{aligned}$$

for all $Y \in L^\infty(\mathcal{F})$ with $Y \geq 0$ and $n \in \mathbb{N}$. Hence, $\mu \notin \mathrm{dom} f^*$ if $\mu Y < -Y$ with positive probability.

(iv) Let $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$. Since f is cash invariant we derive for all $Y \in L^\infty(\mathcal{F})$

$$f^*(\mu) \geq \mu Y - f(Y) = \mu Y + Y - f(0).$$

This implies that $\mu \in \mathrm{dom} f^*$ only if $\mu Y = -Y$ for all $Y \in L^\infty(\mathcal{F})$; whence $-\mu$ is a projection. $\square$

We derive a variant of Proposition 2.5 for convex functions.

Proposition 2.10. *A continuous convex function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is local if and only if* $\mathrm{dom} f^* \subset \{Z \in L^q(\mathcal{E}) \mid E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})\}$. *Moreover, in this case*

$$f(X) = \operatorname*{ess.sup}_{Z \in L^q(\mathcal{E}), E[|Z|^q|\mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{9}$$

Proof. In view of Theorem 2.4 holds $f(X) = \operatorname{ess.sup}_{\mu \in \mathrm{dom} f^*}(\mu X - f^*(\mu))$ for all $X \in L^p(\mathcal{E})$. In case that f is local, it follows from Lemma 2.9 (i) that any $\mu \in \mathrm{dom} f^*$ is local and in view of Proposition 2.5 it is of the form (5) for some representing $Z \in L^q(\mathcal{E})$ with $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$. Conversely, any function of the form (9) is local. $\square$

In the same manner, we can derive from Lemma 2.9 (ii) and (iii) an analogue of Proposition 2.6, for convex functions.

Proposition 2.11. *A continuous convex function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *monotone and*

(ii) *subcash invariant*

if and only if $\mathrm{dom} f^* \subset \mathcal{C} = \{Z \in L^q(\mathcal{E}) \mid E[Z \mid \mathcal{F}] \geq -1, Z \leq 0, E[|Z|^q \mid \mathcal{F}] \in L^{r(p-1)/(p-r)}(\mathcal{F})\}$. *Moreover, in this case*

$$f(X) = \underset{Z \in \mathcal{C}}{\mathrm{ess.sup}}(E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{10}$$

Remark 2.12. *We obtain the convex variant of Remark 2.7; that is, in Proposition 2.11 we can replace subcash invariance by cash invariance and then write* $E[Z \mid \mathcal{F}] = -1$ *in place of* $E[Z \mid \mathcal{F}] \geq -1$ *in the definition of* $\mathcal{C}$.

2.4 Conditional mean variance

In this section, we let $p = 4$, $r = 2$ and fix $\beta \in \mathbb{R}$, $\beta > 0$. The conditional mean variance $f : L^4(\mathcal{E}) \to L^2(\mathcal{F})$ is defined by

$$f(X) = E[-X \mid \mathcal{F}] + \frac{\beta}{2} Var[X \mid \mathcal{F}],$$

where $Var[X \mid \mathcal{F}] = E[X^2 \mid \mathcal{F}] - E[X \mid \mathcal{F}]^2$ denotes the conditional variance of $X \in L^4(\mathcal{E})$.

Based on the following lemma, we explicitly construct a subgradient of f.

Lemma 2.13. *Let* $f : L^4(\mathcal{E}) \to L^2(\mathcal{F})$ *denote the conditional mean variance. Then,*

$$\mathrm{dom} f^* = \{Z \in L^{\frac{4}{3}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] = -1, E[|Z|^{\frac{4}{3}} \mid \mathcal{F}] \in L^3(\mathcal{F})\}. \tag{11}$$

Moreover, for all $Z \in \mathrm{dom} f^*$

$$f^*(Z) = \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]$$

and, in particular, $(1+Z)/\beta \in \partial f^*(Z)$.

Proof. Inspection shows that the conditional mean variance is cash invariant continuous convex and local. Thus, by Lemma 2.9 (i) and (iv) $-\mu$ is local and a projection for each $\mu \in \mathrm{dom} f^*$ which proves the inclusion "$\subset$" in (11).

To prove the reverse inclusion, let $Z \in L^{\frac{4}{3}}(\mathcal{E})$ with $E[Z \mid \mathcal{F}] = -1$ and $E[|Z|^{\frac{4}{3}} \mid \mathcal{F}] \in L^3(\mathcal{F})$. We will show that $f^*(Z) = \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]$. To this end, observe

$$\begin{aligned} f^*(Z) &= \underset{X \in L^4(\mathcal{E})}{\mathrm{ess.sup}} (E[ZX \mid \mathcal{F}] - f(X)) \\ &= \underset{X \in L^4(\mathcal{E})}{\mathrm{ess.sup}} \left(E[(1+Z)X \mid \mathcal{F}] - \frac{\beta}{2} Var[X \mid \mathcal{F}] \right) \\ &= \underset{X \in L^4(\mathcal{E}), E[X \mid \mathcal{F}]=0}{\mathrm{ess.sup}} \left(E[(1+Z)X \mid \mathcal{F}] - \frac{\beta}{2} Var[X \mid \mathcal{F}] \right) \\ &= \underset{X \in L^4(\mathcal{E}), E[X \mid \mathcal{F}]=0}{\mathrm{ess.sup}} E\left[(1+Z)X - \frac{\beta}{2} X^2 \mid \mathcal{F} \right]. \end{aligned} \tag{12}$$

An element $X' \in L^4(\mathcal{E})$ which satisfies the first order condition

$$1 + Z - \beta X' = 0 \tag{13}$$

is necessarily a pointwise maximizer of the integrands $(1+Z)X - \frac{\beta}{2}X^2$ in (12) (maximized over all of $L^4(\mathcal{E})$). In view of (13) we therefore define the maximizer $X^* = (1+Z)/\beta$; fortunately, $E[X^* \mid \mathcal{F}] = 0$. Plugging X^* into (12) yields the assertion. $\square$

Combining (3) and Lemma 2.13 we conclude: if $Z^* \in \mathcal{L}(L^4(\mathcal{E}), L^2(\mathcal{E}))$ maximizes

$$\begin{aligned} f(X) &= \operatorname*{ess.sup}_{Z \in \mathcal{C}} (E[ZX \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathcal{C}} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] \right) \end{aligned} \tag{14}$$

that is

$$f(X) = E[Z^* X \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z^*)^2 \mid \mathcal{F}]$$

for some $X \in L^4(\mathcal{E})$, then $Z^* \in \partial f(X)$.

Theorem 2.14. *Let $f : L^4(\mathcal{E}) \to L^2(\mathcal{F})$ denote the conditional mean variance. Then, for all $X \in L^4(\mathcal{E})$*

$$\beta(X - E[X \mid \mathcal{F}]) - 1 \in \partial f(X).$$

Proof. Let $X \in L^4(\mathcal{E})$. Since $f(X - E[X \mid \mathcal{F}]) = f(X) + E[X \mid \mathcal{F}]$ we have $\partial f(X - E[X \mid \mathcal{F}]) = \partial f(X)$. If $Z' \in \mathcal{L}(L^4(\mathcal{E}), L^2(\mathcal{F}))$ satisfies the first order condition

$$X - E[X \mid \mathcal{F}] - \frac{1}{\beta}(1 + Z') = 0 \tag{15}$$

then Z' is necessarily a pointwise maximizer of the integrands

$$Z(X - E[X \mid \mathcal{F}]) - \frac{1}{2\beta}(1+Z)^2$$

in (14) (adjusted for $-E[X \mid \mathcal{F}]$ and maximized over all of $L^4(\mathcal{E})$). In view of (15) we therefore define the maximizer $Z^* = \beta(X - E[X \mid \mathcal{F}]) - 1$; fortunately $E[Z^* \mid \mathcal{F}] = -1$ as well as $E[|Z^*|^{\frac{4}{3}} \mid \mathcal{F}] \in L^3(\mathcal{F})$ which means that Z^* maximizes (14). $\square$

To summarize, standard vector space based convex analysis is applicable to a selected class of conditional risk measures. This class contains risk measures which map $L^p(\mathcal{E})$ into $L^r(\mathcal{F})$.

3 The module approach

In this section we follow a module approach to conditional risk measures. We briefly repeat the most important features of L^p type modules, a comprehensive treatment of which can be found in [KV09] and for further background we refer to [FKV09].

3.1 Preliminaries

Unless stated otherwise, we let $p \in [1, +\infty]$ throughout this section. Recall that the classical conditional expectation $E[\cdot \mid \mathcal{F}] : L^1(\mathcal{E}) \to L^1(\mathcal{F})$ extends to the conditional expectation $E[\cdot \mid \mathcal{F}] : L^0_+(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by

$$E[X \mid \mathcal{F}] = \lim_{n \to \infty} E[X \wedge n \mid \mathcal{F}]. \tag{16}$$

We define the function $\|\cdot\|_p : L^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by

$$\|X\|_p = \begin{cases} E[|X|^p \mid \mathcal{F}]^{1/p} & \text{if } p \in [1, \infty) \\ \text{ess.inf}\{Y \in \bar{L}^0_+(\mathcal{F}) \mid Y \geq |X|\} & \text{if } p = \infty \end{cases}, \tag{17}$$

and

$$L^p_{\mathcal{F}}(\mathcal{E}) = \{X \in L^0(\mathcal{E}) \mid \|X\|_p \in L^0(\mathcal{F})\}.$$

The standard properties of the conditional expectation guarantee that $\|\cdot\|_p$ is an $L^0(\mathcal{F})$–norm on $L^p_{\mathcal{F}}(\mathcal{E})$, that is, $\|\cdot\|_p : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0_+(\mathcal{F})$ satisfies

(i) $\|X\|_p = 0$ if and only if $X = 0$,

(ii) $\|YX\|_p = |Y|\|X\|_p$ for all $Y \in L^0(\mathcal{F})$ and $X \in L^p_{\mathcal{F}}(\mathcal{E})$,

(iii) $\|X + X'\|_p \leq \|X\|_p + \|X'\|_p$ for all $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$.

We endow $L^p_{\mathcal{F}}(\mathcal{E}) = (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ with the module topology induced by the $L^0(\mathcal{F})$–norm $\|\cdot\|_p$ and we endow $L^0(\mathcal{F}) = (L^0(\mathcal{F}), |\cdot|)$ with the ring topology induced by the absolute value $|\cdot|$. Then $L^p_{\mathcal{F}}(\mathcal{E})$ becomes a topological $L^0(\mathcal{F})$–module over the topological ring $L^0(\mathcal{F})$. For further details we refer to [FKV09, KV09]. We work with the convention that the conditional expectation $E[\cdot \mid \mathcal{F}] : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is understood as

$$E[X \mid \mathcal{F}] = E[X^+ \mid \mathcal{F}] - E[X^- \mid \mathcal{F}],$$

the right hand side of which is understood as in (16).

Example 3.1. *Let us assume that $\mathcal{F} = \sigma(A_1, \ldots, A_m)$ is generated by a finite partition $A_1, \ldots, A_m$ of Ω.*

The local structure, formerly a property in reference to the functions we studied, now also appears as a property of the model spaces $L^p_{\mathcal{F}}(\mathcal{E})$ in the sense that on each $\mathcal{F}$–atom A_i, $1 \leq i \leq m$, we consider a classical L^p space, namely

$L^p(\mathcal{E}\cap A_i) = L^p(\Omega\cap A_i, \mathcal{E}\cap A_i, P_i)$, where P_i denotes $P[\cdot \mid A_i]$. Over all of $\mathcal{E}$, these spaces are "pasted" together to become

$$L^p_{\mathcal{F}}(\mathcal{E}) = \sum_{i=1}^m 1_{A_i} L^p(\mathcal{E}\cap A_i).$$

Consequently, if $\mathcal{F}$ is finitely generated, $L^p(\mathcal{E}) = L^p_{\mathcal{F}}(\mathcal{E})$ and no additional structure is provided.

However, if $\mathcal{F}$ is generated by a countable partition (A_n) of Ω then $L^p_{\mathcal{F}}(\mathcal{E})$ becomes

$$L^p_{\mathcal{F}}(\mathcal{E}) = \sum_{n\in\mathbb{N}} 1_{A_n} L^p(\mathcal{E}\cap A_n)$$

which in fact is an $L^0(\mathcal{F})$–module significantly larger than $L^p(\mathcal{E})$. Indeed, it is not hard to see that $X_n \in L^p(\mathcal{E}\cap A_n)$ for all $n\in\mathbb{N}$ is not sufficient for $\sum_{n\in\mathbb{N}} 1_{A_n} X_n \in L^p(\mathcal{E})$ in general.

A function $\mu : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is $L^0(\mathcal{F})$–linear if $\mu(YX+X') = Y\mu X + \mu X'$ for all $Y\in L^0(\mathcal{F})$ and $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$. In (1) we have already defined localness for functions from $L^p(\mathcal{E})$ into $L^r(\mathcal{F})$. We adapt this to functions $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ with the convention $0\cdot(\pm\infty) = 0$.

A function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is proper if $f(X) > -\infty$ for all $X\in L^p_{\mathcal{F}}(\mathcal{E})$ and if there is at least one $X \in L^p_{\mathcal{F}}(\mathcal{E})$ such that $f(X) < +\infty$. We define

$$\begin{aligned} PI(f) &= \text{ess.sup}\{A\in\mathcal{F} \mid 1_A f = 1_A(+\infty)\} \\ MI(f) &= \text{ess.sup}\{A\in\mathcal{F} \mid \exists X\in L^p_{\mathcal{F}}(\mathcal{E}) : 1_A f(X) = 1_A(-\infty)\} \\ R(f) &= (PI(f)\cup MI(f))^c \end{aligned}$$

so that f is proper on $R(f)$, $f\equiv +\infty$ on $PI(f)$ and f may take the value $-\infty$ on $MI(f)$. The effective domain $\text{dom}f$ of f is defined by

$$\text{dom}f = \{X\in L^p_{\mathcal{F}}(\mathcal{E}) \mid 1_{PI(f)^c} f(X) < +\infty\}. \tag{18}$$

Trivially, $P[PI(f)\cap MI(f)\cap R(f)] = 0$ so that f is proper only if $P[PI(f)] = P[MI(f)] = 0$. If f is local then we even have "if and only if".

In [FKV09, KV09] $L^0(\mathcal{F})$–convexity is only defined for proper functions. For the purposes of Section 4 below in which we use dual techniques to construct hulls of proper $L^0(\mathcal{F})$–convex functions, we have to extend this definition in a consistent way to functions that are not proper.

In vector space theory one agrees on the convention that $-\infty+\infty = +\infty$ and defines a function $f : V\to[-\infty,+\infty]$ on a real vector space V to be convex if $f(\alpha v + (1-\alpha)w) \le \alpha f(v) + (1-\alpha)f(w)$ for all $v, w\in V$, $\alpha\in[0,1]$. In line with this, we set $-\infty+\infty = +\infty$ and define $L^0(\mathcal{F})$–convexity as follows.

Definition 3.2. *A function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is $L^0(\mathcal{F})$–convex if*

$$f(YX + (1-Y)X') \le Yf(X) + (1-Y)f(X')$$

for all $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y\in L^0(\mathcal{F})$ with $0\le Y\le 1$. (Recall the convention $0\cdot(\pm\infty) = 0$.)

Remark 3.3. *Inspection shows that a function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *is* $L^0(\mathcal{F})$*–convex if and only if for all* $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$ *and* $Y \in L^0(\mathcal{F})$ *with* $0 \leq Y \leq 1$,

$$f(YX + (1-Y)X') \leq Yf(X) + (1-Y)f(X') \tag{19}$$

on the set $(\{f(X) = -\infty, f(X') = +\infty\} \cup \{f(X) = +\infty, f(X') = -\infty\})^c$.

Lemma 3.4. *Any* $L^0(\mathcal{F})$*–convex function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *is local.*

Proof. Let $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $A \in \mathcal{F}$. Then, we derive the inequalities

$$\begin{aligned} f(1_A X) &\leq 1_A f(X) + 1_{A^c} f(0) \\ &= 1_A f(1_A(1_A X) + 1_{A^c} X) + 1_{A^c} f(0) \\ &\leq 1_A f(1_A X) + 1_{A^c} f(0) \end{aligned}$$

which become equalities on multiplying by 1_A. □

Consider a local function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$. As in [FKV09], we call f lower semicontinuous (l.s.c.) if for any convergent net $X_N \to X$ in $L^p_{\mathcal{F}}(\mathcal{E})$ we have

$$\underset{N}{\text{ess.liminf}}\, f(X_N) \geq f(X),$$

where we define $\text{ess.liminf} Y_N = \text{ess.sup}_N \text{ess.inf}_{M \geq N} Y_M$ for a net (Y_N) in $L^p_{\mathcal{F}}(\mathcal{E})$.

Definition 3.5. *Let* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *be a local function. The closure* $\text{cl}(f) : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *of* f *is given by*

$$\text{cl}(f) = 1_{MI(f)^c} g + 1_{MI(f)}(-\infty),$$

where g *is the greatest l.s.c.* $L^0(\mathcal{F})$*–convex function majorized by* $1_{MI(f)^c} f$*. The function* f *is closed if* $f = \text{cl}(f)$*.*

By definition, $\text{cl}(f)$ is l.s.c. $L^0(\mathcal{F})$–convex and in particular local. By definition, a closed local function is $L^0(\mathcal{F})$–convex.

For $p \in [1, +\infty)$ we have the following analogy to (2). Any continuous $L^0(\mathcal{F})$–linear function $\mu : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is of the form

$$\mu X = E[ZX \mid \mathcal{F}] \tag{20}$$

for some $Z \in L^q_{\mathcal{F}}(\mathcal{E})$, where $q = p/(p-1)$ if $p \in (1, \infty)$ and $q = \infty$ if $p = 1$, cf. [KV09]. The conjugate function $f^* : L^q_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of a local function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is defined by

$$f^*(Z) = \underset{X \in L^p_{\mathcal{F}}(\mathcal{E})}{\text{ess.sup}} (E[ZX \mid \mathcal{F}] - f(X)) = \underset{X \in \text{dom} f}{\text{ess.sup}} (E[ZX \mid \mathcal{F}] - f(X))$$

and the conjugate $f^{**} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of f^* is defined by

$$f^{**}(X) = \underset{Z \in L^q_{\mathcal{F}}(\mathcal{E})}{\text{ess.sup}} (E[ZX \mid \mathcal{F}] - f^*(Z)) = \underset{Z \in \text{dom} f^*}{\text{ess.sup}} (E[ZX \mid \mathcal{F}] - f^*(Z)), \tag{21}$$

where the second equality follows from the definition of the effective domain in (18). The next theorem presents an $L^0(\mathcal{F})$–convex duality relation which slightly generalizes the Fenchel–Moreau type dual representation of Theorem 3.8 in [FKV09].

Theorem 3.6. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a local function. Then,*

$$f^{**} = \mathrm{cl}(f).$$

In particular, if f is proper l.s.c. $L^0(\mathcal{F})$–convex then $f = f^{**}$.

Proof. We first prove the auxiliary claim that an $L^0(\mathcal{F})$–convex l.s.c. function $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ with $g > -\infty$ satisfies the $L^0(\mathcal{F})$–convex duality relation

$$g = g^{**} \tag{22}$$

which proves the second statement. Indeed, let $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and define $A = \{g(X) < +\infty\}$. Then, on A^c the relation (22) is trivially valid for X. To see that (22) is also valid for X on A it suffices to observe that $1_A g$ is a proper $L^0(\mathcal{F})$–convex l.s.c. function and to apply Theorem 3.8 in [FKV09] by which $1_A g = (1_A g)^{**}$. Since g is local by $L^0(\mathcal{F})$–convexity we conclude

$$g(X) = 1_A g(X) + 1_{A^c} g(X) = (1_A g)^{**}(X) + (1_{A^c} g)^{**}(X) = g^{**}(X)$$

which proves the auxiliary claim.

Next, define $f_1 = 1_{MI(f)^c} f$ and $f_2 = 1_{MI(f)} f$. We show separately that

$$f_1^{**} = \mathrm{cl}(f_1) \text{ and } f_2^{**} = \mathrm{cl}(f_2)$$

which by localness of f^{**} and $\mathrm{cl}(f)$ yields the assertion.

To see that $f_1^{**} = \mathrm{cl}(f_1)$ observe that by definition f_1^{**} is $L^0(\mathcal{F})$–convex l.s.c. and $-\infty < f_1^{**} \le f_1$. Further, from

$$\mathrm{cl}(f_1) \le f_1 \text{ implies } \mathrm{cl}(f_1)^* \ge f_1^* \text{ implies } \mathrm{cl}(f_1) = \mathrm{cl}(f_1)^{**} \le f_1^{**}$$

we derive $f_1^{**} = \mathrm{cl}(f_1)$.

To establish $f_2^{**} = \mathrm{cl}(f_2)$ we show that there is some $X_{-\infty} \in L^p_{\mathcal{F}}(\mathcal{E})$ with $f_2(X_{-\infty}) = 1_{MI(f)}(-\infty)$. Indeed, since f is local the collection

$$\mathcal{S} = \{A \in \mathcal{F} \mid \exists X \in L^p_{\mathcal{F}}(\mathcal{E}) : f(X) = -\infty \text{ on } A\}$$

is directed upwards and by definition we have $\text{ess.sup}\mathcal{S} = MI(f)$. Hence, there exists an increasing sequence $(A_n) \subset \mathcal{F}$ and a corresponding sequence (X_n) in $L^p_{\mathcal{F}}(\mathcal{E})$ with $A_n \nearrow M_{-\infty}$ and $f(X_n) = -\infty$ on A_n for each $n \in \mathbb{N}$. Since f is local

$$X_{-\infty} = \sum_{i=1}^{\infty} 1_{A_i \setminus \bigcup_{j=1}^{i-1} A_j} X_i$$

is as required with $A_0 = \emptyset$. We conclude that

$$\begin{aligned} f_2^* &= \underset{X \in L^p_{\mathcal{F}}(\mathcal{E})}{\text{ess.sup}} \left(E[\cdot X \mid \mathcal{F}] - f_2(X)\right) \\ &\ge E[\cdot X_{-\infty} \mid \mathcal{F}] - f_2(X_{-\infty}) \ge 1_{MI(f)}(+\infty). \end{aligned}$$

This together with (22) and localness of f implies $f_2^{**} = 1_{MI(f)}(-\infty) = \mathrm{cl}(f_2)$. (Note, that $MI(f) = MI(f_2)$.) $\square$

Remark 3.7. *The epigraph* $\mathrm{epi}f = \{(X,Y) \in L^p_{\mathcal{F}}(\mathcal{E}) \times L^0(\mathcal{F}) \mid f(X) \le Y\}$ *of a closed function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *is closed w.r.t. the product topology. To see this, it suffices to observe that* $1_{MI(f)^c}\mathrm{epi}f_1$ *is closed cf. [FKV09] and that* $1_{MI(f)}\mathrm{epi}f_2 = 1_{MI(f)}(L^p_{\mathcal{F}}(\mathcal{E}) \times L^0(\mathcal{F}))$ *is closed as well;* f_1 *and* f_2 *are understood as in the above proof. Since* $MI(f)$ *and* $MI(f)^c$ *are disjoint the sum of the two* $1_{MI(f)^c}\mathrm{epi}f_1 + 1_{MI(f)}(L^p_{\mathcal{F}}(\mathcal{E}) \times L^0(\mathcal{F})) = \mathrm{epi}f$ *is also closed.*

Lemma 3.8. *Let* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *be a local function. Then,*

$$PI(f) \subset MI(f^*) \text{ and } MI(f) \subset PI(f^*). \tag{23}$$

If f *is closed* $L^0(\mathcal{F})$*–convex we have equalities.*

Proof. Since f is local (23) follows from the definitions of $PI(\cdot), MI(\cdot)$ and f^*. On replacing f with f^* the reverse inclusions follow as for closed $L^0(\mathcal{F})$–convex f we have $f = f^{**}$, cf. Theorem 3.6. □

The preceding lemma reveals in particular that for a closed $L^0(\mathcal{F})$–convex function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ we have the following decompositions

$$f^* = 1_{PI(f)}(-\infty) + 1_{MI(f)}(+\infty) + 1_{R(f)}f^* \tag{24}$$

$$f = f^{**} = 1_{PI(f)}(+\infty) + 1_{MI(f)}(-\infty) + 1_{R(f)}f^{**}. \tag{25}$$

Definition 3.9. *Let* $p \in [1, +\infty)$, q *be as above and* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *be a proper function. An element* $Z \in L^q_{\mathcal{F}}(\mathcal{E})$ *is a subgradient of a* f *at* $X_0 \in \mathrm{dom}f$ *if*

$$E[Z(X - X_0) \mid \mathcal{F}] \le f(X) - f(X_0), \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E}).$$

The set of all subgradients of f *at* X_0 *is denoted by* $\partial f(X_0)$.

Example 3.10. *Let* $\mathcal{F} = \sigma(A_1, A_2, A_3)$ *be finitely generated, where* $(A_i)_{1 \le i \le 3} \subset \mathcal{E}$ *is pairwise disjoint with* $P[A_i] > 0$, $1 \le i \le 3$ *and* $\Omega = \bigcup_{i=1}^3 A_i$. *We consider a function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *and we identify* $\bar{L}^0(\mathcal{F})$ *with* $(\mathbb{R} \cup \{\pm\infty\})^3$ *so that* $f = (f_1, f_2, f_3)$ *for three functions* $f_1, f_2, f_3 : L^p_{\mathcal{F}}(\mathcal{E}) \to [-\infty, +\infty]$. *Let us further assume that* $f_1 \equiv +\infty$, f_2 *is proper and there exists* $X \in L^p_{\mathcal{F}}(\mathcal{E})$ *such that* $f_3(X) = -\infty$.

Then $PI(f) = A_1$ *and* $MI(f) = A_3$. *Further,* $X \in \mathrm{dom}f$ *if and only if* $f_2(X), f_3(X) < +\infty$ *irrespectively of the fact that* $f_1(X) = +\infty$. *The function* f *would be proper if and only if* f_1, f_2 *and* f_3 *were proper at the same time. Thus,* $1_{A_2}f$ *is proper while* f *is not. In the same way we see that* f *is* $L^0(\mathcal{F})$*–convex if and only if each* f_i *is convex,* $1 \le i \le 3$.

If, in addition, f *is local then we can identify* f *with three functions* $f_1, f_2, f_3 : L^p(\mathcal{E} \cap A_i) \to [-\infty, +\infty]$ *defined on classical* L^p *spaces. Then* f *is l.s.c. if and only if each* f_i *is l.s.c.,* $1 \le i \le 3$, *and its closure is given by*

$$\mathrm{cl}(f) = (\mathrm{cl}(f_1), \mathrm{cl}(f_2), \mathrm{cl}(f_3)) = (+\infty, f_2^{**}, -\infty).$$

The main advantage of the module approach over the vector space approach from Section 2 is the fact that we can consider conditional risk measures on $L^p_{\mathcal{F}}(\mathcal{E})$ which is a much larger model space than $L^p(\mathcal{E})$. Furthermore, within the module approach, duality results are applicable to functions which may take values in $\bar{L}^0(\mathcal{F})$. As a consequence, examples such as the conditional entropic risk measure is fully covered.

Further, within the vector space approach, continuous linear functions $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ are not necessarily conditional expectations. One has to employ the results of Section 2.2 to derive that only those continuous linear functions which are conditional expectations are relevant for conditional risk measures.

In contrast to this, continuous $L^0(\mathcal{F})$–linear functions from $L^p_{\mathcal{F}}(\mathcal{E})$ into $L^0(\mathcal{F})$ are conditional expectations as stated in (20). Results analogous to Proposition 2.5, Proposition 2.6 and Remark 2.7 presented in Section 2.2 are not required. In this sense, the module approach provides us a priori with an interpretation of (21) in terms of expected losses under different scenarios which, by virtue of f^*, are taken more or less seriously.

3.2 Monotone (sub)cash invariant $L^0(\mathcal{F})$–convex functions on $L^p_{\mathcal{F}}(\mathcal{E})$

In this section, we fix $p \in [1, +\infty)$ and define q dual to p, as usual. The next definition is similar to that of 2.1. However, as we work in a module setup, a few amendments are needed.

Definition 3.11. *A function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$*

(i) *is monotone if $f(X) \leq f(X')$ for all $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$ with $X \geq X'$,*

(ii) *is subcash invariant if $f(X + Y) \geq f(X) - Y$ for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_+(\mathcal{F})$,*

(iii) *is cash invariant if $f(X + Y) = f(X) - Y$ for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0(\mathcal{F})$.*

A set $\mathcal{P} \subset L^p_{\mathcal{F}}(\mathcal{E})$ is $L^0(\mathcal{F})$–convex if $YX + (1-Y)X' \in \mathcal{P}$ whenever $X, X' \in \mathcal{P}$ and $Y \in L^0(\mathcal{F})$ with $0 \leq Y \leq 1$. The epigraph of an $L^0(\mathcal{F})$–convex function is $L^0(\mathcal{F})$–convex. $\mathcal{P}$ is an $L^0(\mathcal{F})$–cone if $YX \in \mathcal{P}$ for all $X \in \mathcal{P}$ and $Y \in L^0_+(\mathcal{F})$. For the same reasons as in the vector space case we refer to $L^0(\mathcal{F})$–convex functions $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ which are monotone and cash invariant as conditional risk measures.

From now on, let $\mathcal{P} = \{X \in L^p_{\mathcal{F}}(\mathcal{E}) \mid X \geq 0\}$ and observe that $\mathcal{P}$ is a closed $L^0(\mathcal{F})$–convex $L^0(\mathcal{F})$–cone. $\mathcal{P}$ induces the partial order of almost sure dominance on $L^p_{\mathcal{F}}(\mathcal{E})$ via

$$X \geq X' \Leftrightarrow X - X' \in \mathcal{P}.$$

Inspection shows that $(L^p_{\mathcal{F}}(\mathcal{E}), \geq)$ is an ordered module, cf. [KV09]. The polar $L^0(\mathcal{F})$–cone $\mathcal{P}^\circ$ of $\mathcal{P}$ is

$$\mathcal{P}^\circ = \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid \forall X \in \mathcal{P} : E[ZX \mid \mathcal{F}] \leq 0\}.$$

Inspection shows that $\mathcal{P}^\circ = \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid Z \leq 0\}$ by definition of $\mathcal{P}$. Further, define

$$\begin{aligned} s\mathcal{D} =& \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] \geq -1\} \\ \mathcal{D} =& \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] = -1\}. \end{aligned}$$

Note that if $Z \in s\mathcal{D}$ then $E[-ZY \mid \mathcal{F}] \leq Y$ and if $Z \in \mathcal{D}$ then $E[-ZY \mid \mathcal{F}] = Y$ for all $Y \in L^0(\mathcal{F})$. The next proposition is a variant of the bipolar theorem, for modules.

Proposition 3.12. *Let $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$. Then $X \geq X'$ if and only if $E[Z(X - X') \mid \mathcal{F}] \leq 0$ for all $Z \in \mathcal{P}^\circ$.*

Proof. This follows from the corresponding definitions. $\square$

Lemma 3.13. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a closed $L^0(\mathcal{F})$–convex function.*

(i) *f is monotone if and only if $1_{R(f)}\mathrm{dom} f^* \subset 1_{R(f)}\mathcal{P}^\circ$.*

(ii) *f is subcash–invariant if and only if $1_{R(f)}\mathrm{dom} f^* \subset 1_{R(f)} s\mathcal{D}$.*

(iii) *f is cash–invariant if and only if $1_{R(f)}\mathrm{dom} f^* \subset 1_{R(f)}\mathcal{D}$.*

Proof. Let $X_0 \in L^p_{\mathcal{F}}(\mathcal{E})$ be such that $f(X_0) \in L^0(\mathcal{F})$ on $R(f)$.

(i) To prove the only if statement, assume by way of contradiction that there is $Z \in \mathrm{dom} f^*$ with $P[B \cap R(f)] > 0$, where $B := \{Z > 0\}$. By monotonicity of f we have $f(X_0 + n1_B) \leq f(X_0)$ for all $n \in \mathbb{N}$. Thus, on $R(f)$ holds

$$f^*(Z) \geq E[Z(X_0 + n1_B) \mid \mathcal{F}] - f(X_0 + n1_B) \geq nE[Z1_B \mid \mathcal{F}] + E[ZX_0 \mid \mathcal{F}] - f(X_0)$$

which contradicts $f^*(Z) < +\infty$. To establish the if statement, recall the decompositions (24) and (25). Thus, $1_{R(f)}\mathrm{dom} f^* \subset 1_{R(f)}\mathcal{P}^\circ$ implies

$$\begin{aligned} f(X) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[ZX \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} (E[ZX \mid \mathcal{F}] - f^*(Z)), \end{aligned}$$

for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$. Hence, by Proposition 3.12, f is monotone.

(ii) To prove the only if statement, let $Z \in \mathrm{dom} f^*$ and assume that $P[\{E[Z \mid \mathcal{F}] < -1\} \cap R(f)] > 0$. By subcash invariance of f,

$$\begin{aligned} f^*(Z) &\geq \operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f(X + nY) - nY) \\ &\overset{X' = X + nY}{=} \operatorname*{ess.sup}_{X' \in L^p_{\mathcal{F}}(\mathcal{E})} (E[Z(X' - nY) \mid \mathcal{F}] - f(X') - nY) \\ &= \operatorname*{ess.sup}_{X' \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX' \mid \mathcal{F}] - f(X') - nY(E[Z \mid \mathcal{F}] + 1)) \\ &= f^*(Z) - nY(E[Z \mid \mathcal{F}] + 1) \end{aligned}$$

for all $Y \in L^0_+(\mathcal{F})$ and $n \in \mathbb{N}$ which contradicts $f^*(Z) < +\infty$ on $R(f)$. To establish the if statement, observe that the decompositions in (24) and (25) together with $1_{R(f)}\mathrm{dom} f^* \subset 1_{R(f)} s\mathcal{D}$ imply

$$\begin{aligned} f(X+Y) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in s\mathcal{D}} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \geq f(X) - Y \end{aligned}$$

for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_+(\mathcal{F})$.

(iii) To prove the only if statement, assume that there is $Z \in \mathrm{dom} f^*$ with $P[\{E[Z \mid \mathcal{F}] \neq -1\} \cap R(f)] > 0$. Since f is cash invariant we derive for all $Y \in L^0(\mathcal{F})$

$$f^*(Z) \geq E[Z(X_0+Y) \mid \mathcal{F}] - f(X_0+Y) = Y(E[Z \mid \mathcal{F}]+1) + E[ZX_0 \mid \mathcal{F}] - f(X_0).$$

This contradicts $f^*(Z) < +\infty$ on $R(f)$. Conversely, to establish the if statement, let $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0(\mathcal{F})$. From the decompositions (24) and (25) together with $1_{R(f)}\mathrm{dom} f^* \subset 1_{R(f)}\mathcal{D}$ we derive

$$\begin{aligned} f(X+Y) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathcal{D}} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) = f(X) - Y. \end{aligned}$$

□

Two immediate consequences are the following representation results for monotone subcash invariant $L^0(\mathcal{F})$–convex functions and conditional risk measures.

Corollary 3.14. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0$ be proper l.s.c. $L^0(\mathcal{F})$–convex.*

(i) *If f is monotone and subcash invariant, then for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$*

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ \cap s\mathcal{D}} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{26}$$

(ii) *If f is monotone and cash invariant, then for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$*

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ \cap \mathcal{D}} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{27}$$

Elements of $\mathcal{P}^\circ \cap \mathcal{D}$ can be viewed as transition densities which serve as probabilistic models relative to the initial information $\mathcal{F}$ and uncertain future events $\mathcal{E}$. In this sense, the economic interpretation of static risk measures is preserved under assuming non trivial initial information.

4 Monotone and (sub)cash invariant hulls

4.1 Indicator and support functions

Let $C \subset L^p_{\mathcal{F}}(\mathcal{E})$ be an $L^0(\mathcal{F})$–convex set. We define the mapping $M(\cdot \mid C) : L^p_{\mathcal{F}}(\mathcal{E}) \to \mathcal{F}$ by

$$M(X \mid C) = \text{ess.sup}\{A \in \mathcal{F} \mid 1_A X \in 1_A C\}.$$

The set C has the closure property if for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$

$$1_{M(X|C)} X \in 1_{M(X|C)} C,$$

cf. [FKV09]. The closure property should not be seen as a property in reference to the topology of $L^p_{\mathcal{F}}(\mathcal{E})$. In fact, if $0 \in C$ (which implies that $1_A C \subset C$ for all $A \in \mathcal{F}$) the closure property is closely related to order completeness as it states that a family $(1_A X)_A \subset C$ has a least upper bound in C, namely $\text{ess.sup}_A 1_A X = 1_{M(X|C)} X$.

From now on we assume that C has the closure property. The indicator function $\delta(\cdot \mid C) : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ of C is defined by

$$\delta(X \mid C) = \begin{cases} 0 & \text{on } M(X \mid C) \\ +\infty & \text{on } M(X \mid C)^c \end{cases}.$$

By the closure property of C, $\text{epi}\delta(\cdot \mid C) = C \times L^0_+(\mathcal{F})$. A proper local function is l.s.c. if and only if its epigraph is closed, cf. Proposition 3.4 in [FKV09]. Thus, $\delta(\cdot \mid C)$ is l.s.c. if and only if C is closed.

The support function $\delta^*(\cdot \mid C) : L^q_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of C is defined by

$$\delta^*(Z \mid C) = \underset{X \in C}{\text{ess.sup}}\, E[ZX \mid \mathcal{F}].$$

Since C is $L^0(\mathcal{F})$–convex (in particular $1_A X + 1_{A^c} X' \in C$ for all $A \in \mathcal{F}$ whenever $X, X' \in C$) the support function of C coincides with the conjugate of the indicator function $\delta(\cdot \mid C)$, i.e. for all $Z \in L^q_{\mathcal{F}}(\mathcal{E})$

$$\underset{X \in L^p_{\mathcal{F}}(\mathcal{E})}{\text{ess.sup}} \left(E[ZX \mid \mathcal{F}] - \delta(X \mid C)\right) = \underset{X \in C}{\text{ess.sup}}\, E[ZX \mid \mathcal{F}]. \tag{28}$$

Note that this is also the case if $C = \emptyset$. (28) justifies the notation $\delta^*(\cdot \mid C)$ of the support function.

We define $\delta^{**}(\cdot \mid C) : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ as the conjugate of the support function, i.e.

$$\delta^{**}(X \mid C) = \underset{Z \in L^p_{\mathcal{F}}(\mathcal{E})}{\text{ess.sup}} \left(E[ZX \mid \mathcal{F}] - \delta^*(Z \mid C)\right).$$

If C is closed, we have

$$\delta(\cdot \mid C) = \delta^{**}(\cdot \mid C). \tag{29}$$

Lemma 4.1. *Let $\mathcal{P} = \{X \in L^p_{\mathcal{F}}(\mathcal{E}) \mid X \geq 0\}$ be the order inducing $L^0(\mathcal{F})$–cone and $\mathcal{P}^\circ$ its polar $L^0(\mathcal{F})$–cone. Then*

$$\delta(\cdot \mid \mathcal{P}) = \delta^*(\cdot \mid \mathcal{P}^\circ) \tag{30}$$

$$\delta^*(\cdot \mid \mathcal{P}) = \delta(\cdot \mid \mathcal{P}^\circ) \tag{31}$$

$$\delta^*(X \mid \mathcal{D}) = \begin{cases} -X & \text{on } M(X \mid L^0(\mathcal{F})) \\ \infty & \text{on } M(X \mid L^0(\mathcal{F}))^c \end{cases} \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E}). \tag{32}$$

Proof. To see (30), recall that $\delta^*(X \mid \mathcal{P}^\circ) = \text{ess.sup}_{Z \in \mathcal{P}^\circ} E[ZX \mid \mathcal{F}]$. Further, $1_{M(X|\mathcal{P})} X \geq 0$ implies $1_{M(X|\mathcal{P})} ZX \leq 0$ for all $Z \in \mathcal{P}^\circ$. Since $M(X \mid \mathcal{P}) \in \mathcal{F}$ and since $\mathcal{P}^\circ$ is an $L^0(\mathcal{F})$–cone we derive

$$1_{M(X|\mathcal{P})} \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E[ZX \mid \mathcal{F}] = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E[1_{M(X|\mathcal{P})} ZX \mid \mathcal{F}] = 0.$$

This proves (30) on $M(X \mid \mathcal{P})$.

By definition of $M(X \mid \mathcal{P})$, $1_A X \notin \mathcal{P}$ for all $A \in \mathcal{F}$ with $P[A] > 0$ and $A \subset M(X \mid \mathcal{P})^c$. Since $\mathcal{P}$ is closed $L^0(\mathcal{F})$–convex Theorem 2.8 in[FKV09] implies that there exists $Z_0' \in L^q_{\mathcal{F}}(\mathcal{E})$ and $\varepsilon \in L^0_{++}(\mathcal{F})$ with

$$E[Z_0' X' \mid \mathcal{F}] + \varepsilon \leq E[Z_0' X \mid \mathcal{F}] \tag{33}$$

on $M(X \mid \mathcal{P})^c$ for all $X' \in \mathcal{P}$. The same is true if Z_0' is replaced by $Z_0 = 1_{M(X|\mathcal{P})} Z_0'$. Since $\mathcal{P}$ is an $L^0(\mathcal{F})$–cone we derive that $E[Z_0 X' \mid \mathcal{F}] \leq 0$ for all $X' \in \mathcal{P}$; whence $Z_0 \in \mathcal{P}^\circ$. Further, since $0 \in \mathcal{P}^\circ$ we derive from (33) that $E[Z_0 X \mid \mathcal{F}] > 0$ on $M(X \mid \mathcal{F})^c$. Thus,

$$1_{M(X|\mathcal{P})^c} \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E[ZX \mid \mathcal{F}] \geq 1_{M(X|\mathcal{P})^c} \operatorname*{ess.sup}_{Y \in L^0_+(\mathcal{F})} Y E[Z_0 X \mid \mathcal{F}] = 1_{M(X|\mathcal{P})^c}(+\infty)$$

as $\mathcal{P}^\circ$ is an $L^0(\mathcal{F})$–cone. This proves (30) on all of Ω.

The identity (31) follows by a dual argument as in (29).

To prove (32) we define $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,

$$f(X) = -1_{M(X|L^0(\mathcal{F}))} X + \infty 1_{M(X|L^0(\mathcal{F}))^c}$$

and show that $f^* = \delta(\cdot \mid \mathcal{D})$. (Note that f is the function on the right hand side of (32).) The identity in (32) then follows from a dual argument since $\mathcal{D}$ has the closure property and is $L^0(\mathcal{F})$–convex closed. By definition of f, we have

$$\begin{aligned} f^*(Z) &= \operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f(X)) \\ &= \operatorname*{ess.sup}_{X \in L^0(\mathcal{F})} (X E[Z \mid \mathcal{F}] + X) \\ &= \operatorname*{ess.sup}_{X \in L^0(\mathcal{F})} X (E[Z \mid \mathcal{F}] + 1) \end{aligned}$$

for all $Z \in L^q_{\mathcal{F}}(\mathcal{E})$. The equality $f^* = \delta(\cdot \mid \mathcal{D})$ now follows from the observation that $M(Z \mid \mathcal{D}) = \{E[Z \mid \mathcal{F}] = -1\}$ for all $Z \in L^q_{\mathcal{F}}(\mathcal{E})$. $\square$

4.2 Hulls

Proposition 4.2. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function.*

(i) *The greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{\mathcal{P}^\circ} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,*

$$f_{\mathcal{P}^\circ}(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ}(E[ZX \mid \mathcal{F}] - f^*(Z)).$$

(ii) *The greatest (sub)cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{(s)\mathcal{D}} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,*

$$f_{(s)\mathcal{D}}(X) = \operatorname*{ess.sup}_{Z \in s\mathcal{D}}(E[ZX \mid \mathcal{F}] - f^*(Z)).$$

(iii) *The greatest monotone (sub)cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{\mathcal{P}^\circ,(s)\mathcal{D}} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,*

$$f_{\mathcal{P}^\circ,(s)\mathcal{D}}(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ \cap s\mathcal{D}}(E[ZX \mid \mathcal{F}] - f^*(Z)).$$

Accordingly, we call $f_{\mathcal{P}^\circ}$, $f_{(s)\mathcal{D}}$ and $f_{\mathcal{P}^\circ,(s)\mathcal{D}}$ the monotone, (sub)cash invariant and monotone (sub)cash invariant hull of f, respectively.

Proof. (i) Monotonicity of $f_{\mathcal{P}^\circ}$ follows from Lemma 3.13 (i) and closeness follows from its definition. Further, $f_{\mathcal{P}^\circ} \leq f^{**} \leq f$. Now let $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a monotone closed $L^0(\mathcal{F})$–convex function with $g \leq f$. By Lemma 3.13 (i), $1_{P(g)}\mathrm{dom} g^* \subset 1_{P(g)}\mathcal{P}^\circ$. Thus, $g^* = g^* + \delta(\cdot \mid \mathcal{P}^\circ) \geq f^* + \delta(\cdot \mid \mathcal{P}^\circ)$. Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function. Since $\mathcal{P}^\circ$ is closed $L^0(\mathcal{F})$–convex and has the closure property $\delta(\cdot \mid \mathcal{P}^\circ)$ is l.s.c. $L^0(\mathcal{F})$–convex and hence

$$(f_{\mathcal{P}^\circ})^* = f^* + \delta(\cdot \mid \mathcal{P}^\circ). \tag{34}$$

Hence, $g = g^{**} \leq f_{\mathcal{P}^\circ}$.

(ii) follows similarly.

(iii) As in (34), one checks that $(f_{\mathcal{P}^\circ,(s)\mathcal{D}})^* = f^* + \delta(\cdot \mid \mathcal{P}^\circ \cap (s)\mathcal{D})$. Now the assertion follows as in (i). □

The next remark provides us with an interpretation of monotone and cash invariant hulls.

Remark 4.3. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be proper $L^0(\mathcal{F})$–convex.*

(i) *Define $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ by*

$$g(X) = \operatorname*{ess.inf}_{X' \in L^p_{\mathcal{F}}(\mathcal{E}), X' \leq X} f(X').$$

Note that g need not be proper. For instance, take $f = E[\cdot \mid \mathcal{F}] : L^1_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$, then $g \equiv -\infty$. Nevertheless, g is $L^0(\mathcal{F})$–convex and monotone *with $g \leq f$, and $g = f$ if and only if f is monotone. Moreover, if g is closed then $g = g^{**} = f_{\mathcal{P}^\circ}$ is the greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by f. Indeed, for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$*

$$g(X) = \underset{X_1, X_2 \in L^p_{\mathcal{F}}(\mathcal{E}), X_1 + X_2 = X}{\text{ess.inf}} (f(X_1) + \delta(X_2 \mid \mathcal{P})).$$

With (31) *of Lemma 4.1 one checks that the conjugate of the right hand side equals $f^* + \delta(\cdot \mid \mathcal{P}^\circ)$. Hence, $g^* = (f_{\mathcal{P}^\circ})^*$ by* (34) *and in turn $g^{**} = f_{\mathcal{P}^\circ}$.*

(ii) *Define $h : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ by*

$$h(X) = \underset{Y \in L^0(\mathcal{F})}{\text{ess.inf}} (f(X - Y) - Y).$$

Then h is $L^0(\mathcal{F})$–convex and cash invariant with $h \leq f$, and $h = f$ if and only if f is cash invariant. Moreover, if h is closed then $h = h^{**} = f_{\mathcal{D}}$ is the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f. Indeed, by Lemma 4.1 (32) *we have for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$*

$$h(X) = \underset{X_1, X_2 \in L^p_{\mathcal{F}}(\mathcal{E}), X_1 + X_2 = X}{\text{ess.inf}} (f(X_1) + \delta^*(X_2 \mid \mathcal{D})).$$

Inspection shows that the dual of the right hand side equals $f^* + \delta(\cdot \mid \mathcal{D})$. As in (34) *we have $(f_{\mathcal{D}})^* = f^* + \delta(\cdot \mid \mathcal{D})$. Hence, $h^* = (f_{\mathcal{D}})^*$ and in turn $h^{**} = f_{\mathcal{D}}$.*

Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function. Since

$$\delta(\cdot \mid \mathcal{P}^\circ) + \delta(\cdot \mid (s)\mathcal{D}) = \delta(\cdot \mid \mathcal{P}^\circ \cap (s)\mathcal{D})$$

we derive

$$f_{\mathcal{P}^\circ, (s)\mathcal{D}} = (f_{\mathcal{P}^\circ})_{(s)\mathcal{D}} = (f_{(s)\mathcal{D}})_{\mathcal{P}^\circ}.$$

Further, note that if for instance f is (sub)cash invariant then $f_{\mathcal{P}^\circ, (s)\mathcal{D}} = f_{\mathcal{P}^\circ}$. However, if f is monotone (sub)cash invariant we only have $f_{\mathcal{P}^\circ, (s)\mathcal{D}} = f^{**} \leq f$ as f need not be closed in general.

5 Examples

5.1 Conditional mean variance as cash invariant hull

In this section, we consider the L^2 type module $L^2_{\mathcal{F}}(\mathcal{E})$ and fix $\beta \in \mathbb{R}$, $\beta > 0$. We define a conditional variant $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ of the $L^2(\mathcal{E})$–(semi)–deviation risk measure by

$$f(X) = E[-X \mid \mathcal{F}] + \frac{\beta}{2} E[X^2 \mid \mathcal{F}].$$

One checks that f is proper $L^0(\mathcal{F})$–convex and by Hölder's inequality in the form of (4.13) in [KV09] f is continuous. Next, we consider the mapping $h : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ defined by

$$h(X) = \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F})} (f(X-Y)-Y) = \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F})} \left(E[-X \mid \mathcal{F}] - \frac{\beta}{2} E[(X-Y)^2 \mid \mathcal{F}] \right). \tag{35}$$

An element $Y' \in L^0(\mathcal{F})$ which satisfies the first order condition

$$\beta(E[X \mid \mathcal{F}] - Y') = 0$$

is necessarily a maximizer of the integrands $E[-X \mid \mathcal{F}] - \frac{\beta}{2} E[(X-Y)^2 \mid \mathcal{F}]$ of the right hand side of (35). Thus, plugging in the maximizer $Y^* = E[X \mid \mathcal{F}]$ we derive that h is of the form

$$h(X) = E[-X \mid \mathcal{F}] + \frac{\beta}{2} Var[X \mid \mathcal{F}],$$

where $Var[X \mid \mathcal{F}] = E[X^2 \mid \mathcal{F}] - E[X \mid \mathcal{F}]^2$ denotes the (generalized) conditional variance of $X \in L^2_{\mathcal{F}}(\mathcal{E})$. From this we derive that h is proper $L^0(\mathcal{F})$–convex continuous and in particular closed. By Remark 4.3 (ii) we therefore know that $h = f_{\mathcal{D}}$ is the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f.

In line with the relevant literature we refer to $f_{\mathcal{D}}$ as conditional mean variance. Since $f_{\mathcal{D}}$ is continuous Theorem 3.7 in [FKV09] implies that $\partial f_{\mathcal{D}}(X) \neq \emptyset$ for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$. In particular, for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$ $f_{\mathcal{D}}$ admits a representation of the form

$$f_{\mathcal{D}}(X) = \operatorname*{ess.sup}_{Z \in L^2_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f^*_{\mathcal{D}}(Z)).$$

In what follows we will construct a subgradient of $f_{\mathcal{D}}$ by means of the following lemmas.

Lemma 5.1. *Let $g : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be a function. If $Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ satisfies*

$$g(X) = E[Z^* X \mid \mathcal{F}] - g^*(Z^*),$$

then $Z^* \in \partial g(X)$.

Proof. By definition,

$$g^*(Z) \geq E[ZX \mid \mathcal{F}] - g(X) \tag{36}$$

for all $X, Z \in L^2_{\mathcal{F}}(\mathcal{E})$. Now, let $X, Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ and assume $g(X) = E[Z^* X \mid \mathcal{F}] - g^*(Z^*)$. Then, (36) implies $g(X) \leq E[Z^* X \mid \mathcal{F}] - E[Z^* X' \mid \mathcal{F}] + g(X')$ for all $X' \in L^2_{\mathcal{F}}(\mathcal{E})$, and hence $Z^* \in \partial g(X)$. □

Lemma 5.2. *Let $f_{\mathcal{D}} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ denote the conditional mean variance. Then,*

$$\operatorname{dom} f^*_{\mathcal{D}} = \{Z \in L^2_{\mathcal{F}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] = -1\}. \tag{37}$$

Moreover, for all $Z \in \mathrm{dom} f_{\mathcal{D}}^*$

$$f_{\mathcal{D}}^*(Z) = \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]$$

and, in particular, $(1+Z)/\beta \in \partial f_{\mathcal{D}}^*(Z)$.

Proof. The conditional mean variance is cash invariant closed $L^0(\mathcal{F})$–convex and $R(f_{\mathcal{D}}) = \Omega$. Hence, Lemma 3.13 (iii) yields the inclusion "$\subset$" in (37).

To prove the reverse inclusion in (37), let $Z \in L^2_{\mathcal{F}}(\mathcal{E})$ with $E[Z \mid \mathcal{F}] = -1$. We will show that $f^*(Z) = \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]$. To this end, observe

$$\begin{aligned} f^*(Z) &= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E})} \left(E[ZX \mid \mathcal{F}] - f(X) \right) \\ &= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E})} \left(E[(1+Z)X \mid \mathcal{F}] - \frac{\beta}{2} Var[X \mid \mathcal{F}] \right) \\ &= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E}), E[X|\mathcal{F}]=0} \left(E[(1+Z)X \mid \mathcal{F}] - \frac{\beta}{2} Var[X \mid \mathcal{F}] \right) \\ &= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E}), E[X|\mathcal{F}]=0} E\left[(1+Z)X - \frac{\beta}{2} X^2 \mid \mathcal{F} \right]. \end{aligned} \tag{38}$$

An element $X' \in L^2_{\mathcal{F}}(\mathcal{E})$ which satisfies the first order condition

$$1 + Z - \beta X^* = 0 \tag{39}$$

is necessarily a pointwise maximizer of the integrands $(1+Z)X - \frac{\beta}{2}X^2$ in (38) (maximized over all of $L^2_{\mathcal{F}}(\mathcal{E})$). In view of (39) we therefore define the maximizer $X^* = (1+Z)/\beta$; fortunately, $E[X^* \mid \mathcal{F}] = 0$. Plugging X^* into (38) yields the assertion. □

Combining lemmas 5.1 and 5.2 we conclude: if $Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ maximizes

$$\begin{aligned} f_{\mathcal{D}}(X) &= \operatorname*{ess.sup}_{Z \in L^2_{\mathcal{F}}(\mathcal{E})} \left(E[ZX \mid \mathcal{F}] - f_{\mathcal{D}}^*(Z) \right) \\ &= \operatorname*{ess.sup}_{Z \in L^2_{\mathcal{F}}(\mathcal{E}), E[Z|\mathcal{F}]=-1} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] \right) \end{aligned} \tag{40}$$

that is

$$f_{\mathcal{D}}(X) = E[Z^* X \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z^*)^2 \mid \mathcal{F}]$$

for some $X \in L^2_{\mathcal{F}}(\mathcal{E})$, then $Z^* \in \partial f_{\mathcal{D}}(X)$.

Theorem 5.3. *Let* $f_{\mathcal{D}} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ *denote the conditional mean variance. Then, for all* $X \in L^2_{\mathcal{F}}(\mathcal{E})$

$$\beta(X - E[X \mid \mathcal{F}]) - 1 \in \partial f_{\mathcal{D}}(X).$$

Proof. Let $X \in L^2_{\mathcal{F}}(\mathcal{E})$. Since $f_{\mathcal{D}}(X - E[X \mid \mathcal{F}]) = f_{\mathcal{D}}(X) + E[X \mid \mathcal{F}]$ we have $\partial f_{\mathcal{D}}(X - E[X \mid \mathcal{F}]) = \partial f_{\mathcal{D}}(X)$. If $Z' \in L^2_{\mathcal{F}}(\mathcal{E})$ satisfies the first order condition

$$X - E[X \mid \mathcal{F}] - \frac{1}{\beta}(1 + Z^*) = 0 \tag{41}$$

then Z' is necessarily a pointwise maximizer of the integrands

$$Z(X - E[X \mid \mathcal{F}]) - \frac{1}{2\beta}(1 + Z)^2$$

in (40) (adjusted for $-E[X \mid \mathcal{F}]$ and maximized over all of $L^2_{\mathcal{F}}(\mathcal{E})$). In view of (41) we define the maximizer $Z^* = \beta(X - E[X \mid \mathcal{F}]) - 1$; fortunately $E[Z^* \mid \mathcal{F}] = -1$ which means that Z^* maximizes (40). □

Example 5.4. *If we let $\mathcal{F} = \sigma(A_n)$ as in Example 3.1 we can nicely relate the preceding results to the static case results presented in [FK07]. More precisely, we can identify $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ with a sequence of static $L^2(\mathcal{E})$–(semi)–deviation risk measures $f = (f_1, f_2, f_3, \ldots)$, where $f_n : L^2(\mathcal{E} \cap A_i) \to \mathbb{R}$ is given by*

$$f_n(X) = E_{P_i}[-X] + \frac{\beta}{2} E_{P_i}[X^2],$$

where $E_{P_i}[\cdot]$ denotes the expectation with respect to the probability measure P_i. As derived above, the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f is given by the conditional mean variance $f_{\mathcal{D}} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ which we can also identify with a sequence of static conditional mean variances $f_{\mathcal{D}} = (f_{1,\mathcal{D}}, f_{2,\mathcal{D}}, f_{3,\mathcal{D}}, \ldots)$, where $f_{n,\mathcal{D}} : L^2(\mathcal{E} \cap A_n) \to \mathbb{R}$ is given by

$$f_{n,\mathcal{D}}(X) = E_{P_n}[-X] + \frac{\beta}{2} Var_{P_n}[X]$$

where $Var_{P_n}[\cdot]$ denotes the variance w.r.t. the probability measure P_n, $n \in \mathbb{N}$. Further, by Theorem 5.3 we know that for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$

$$(\beta(X_1 - E_{P_1}[X_1]) - 1, \beta(X_2 - E_{P_2}[X_2]) - 1, \beta(X_3 - E_{P_3}[X_3]) - 1, \ldots) \in \partial f_{\mathcal{D}}(X),$$

where X_n denotes the restriction of X to $\Omega \cap A_n$ which lies in $L^2(\mathcal{E} \cap A_n)$, $n \in \mathbb{N}$.

Alternatively, we could apply the results of Section 5.3 in [FK07]. According to [FK07] the greatest cash invariant closed convex function majorized by f_n is given by the classical mean variance $f_{n,\mathcal{D}}$ for each $n \in \mathbb{N}$. Consequently, the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by $f = (f_1, f_2, f_3, \ldots)$ must be $f_{\mathcal{D}} = (f_{1,\mathcal{D}}, f_{2,\mathcal{D}}, f_{3,\mathcal{D}}, \ldots)$. In the same way, one could proceed with the subgradient, which however is not computed in [FK07].

5.2 Conditional monotone mean variance as monotone hull

As in the previous section we consider the L^2 type module $L^2_{\mathcal{F}}(\mathcal{E})$ and fix $\beta \in \mathbb{R}$, $\beta > 0$. To ease notation we denote by $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ (in place of

$f_{\mathcal{D}}$) the conditional mean variance as introduced in the previous section. In line with Proposition 4.2 we define the conditional monotone mean variance $f_{\mathcal{P}^\circ} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ as the greatest monotone (cash invariant) closed $L^0(\mathcal{F})$–convex function majorized by f. That is,

$$\begin{aligned} f_{\mathcal{P}^\circ}(X) &= \operatorname*{ess.sup}_{Z\in\mathcal{P}^\circ} \left(E[ZX \mid \mathcal{F}] - f^*(Z)\right) \\ &= \operatorname*{ess.sup}_{Z\in\mathcal{P}^\circ\cap\mathcal{D}} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta}E[(1+Z)^2 \mid \mathcal{F}]\right) \end{aligned} \tag{42}$$

By Theorem 3.2 in [KV09] the conditional monotone mean variance $f_{\mathcal{P}^\circ}$ is continuous and $\partial f_{\mathcal{P}^\circ}(X) \neq \emptyset$ for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$. Again, in what follows, we explicitly construct a subgradient.

Lemma 5.5. *Let $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ and $\alpha : L^2_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be functions such that α represents f in the sense that $f = \operatorname{ess.sup}_{Z\in L^2_{\mathcal{F}}(\mathcal{E})}(E[Z\cdot \mid \mathcal{F}] - \alpha(Z))$. If $Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ satisfies*

$$f(X) = E[Z^*X \mid \mathcal{F}] - \alpha(Z^*)$$

then $Z^* \in \partial f(X)$.

Proof. Since α represents f we have

$$\alpha(Z) \geq E[ZX \mid \mathcal{F}] - f(X). \tag{43}$$

Now, let $X, Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ and assume $f(X) = E[Z^*X \mid \mathcal{F}] - \alpha(Z^*)$. Then, (43) implies $f(X) \leq E[Z^*X \mid \mathcal{F}] - E[Z^*X' \mid \mathcal{F}] + f(X')$ for all $X' \in L^2_{\mathcal{F}}(\mathcal{E})$, hence $Z^* \in \partial f(X)$. $\square$

Lemma 5.6. *For all $X \in L^2_{\mathcal{F}}(\mathcal{E})$ and $Z \in L^0_+(\mathcal{F})$ there exists $Y \in L^0(\mathcal{F})$ such that*

$$E[(X+Y)^- \mid \mathcal{F}] = Z.$$

Proof. Let $X \in L^2_{\mathcal{F}}(\mathcal{E})$, $Z \in L^0_+(\mathcal{F})$ and define

$$Y = \operatorname{ess.sup}\{Y' \in L^0(\mathcal{F}) \mid E[(X+Y')^- \mid \mathcal{F}] \geq Z\}.$$

Then Y is as required. Indeed, observe that the function $L^0(\mathcal{F}) \to L^0_+(\mathcal{F}), Y \mapsto E[(X+Y)^- \mid \mathcal{F}]$, is antitone, that is $E[(X+Y_1)^- \mid \mathcal{F}] \geq E[(X+Y_2)^- \mid \mathcal{F}]$ whenever $Y_1 \leq Y_2$. Further,

$$E[(X-n)^- \mid \mathcal{F}] \nearrow +\infty \text{ a.s.}$$

as n tends to $+\infty$. Thus, there exists $Y' \in L^0(\mathcal{F})$ with $E[(X+Y')^- \mid \mathcal{F}] \geq Z$. Hence $Y \in L^0(\mathcal{F})$ and by construction $E[(X+Y)^- \mid \mathcal{F}] \geq Z$. By way of contradiction, assume that $P[A] > 0$, $A = \{E[(X+Y)^- \mid \mathcal{F}] > Z\}$. Let $Y_n = Y + 1/n$, $n \in \mathbb{N}$. Then

$$E[(X+Y_n)^- \mid \mathcal{F}] \nearrow E[(X+Y)^- \mid \mathcal{F}] \text{ a.s.}$$

Hence, $A_n = \{E[(X+Y_n)^- \mid \mathcal{F}] > Z\} \nearrow A$. Thus, there exists $n_0 \in \mathbb{N}$ with $P[A_{n_0}] > 0$. But then,

$$E[(X + 1_{A^c_{n_0}} Y + 1_{A_{n_0}} Y_{n_0})^- \mid \mathcal{F}] \geq Z$$

and $1_{A^c_{n_0}} Y + 1_{A_{n_0}} Y_{n_0} > Y$ on A_{n_0} in contradiction to the maximality of Y. Thus, $E[(X+Y)^- \mid \mathcal{F}] = Z$. □

Theorem 5.7. *Let $f_{\mathcal{P}^\circ} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ denote the conditional monotone mean variance. For $X \in L^2_{\mathcal{F}}(\mathcal{E})$ let $Y \in L^0(\mathcal{F})$ be such that $E[-\beta(X+Y)^- \mid \mathcal{F}] = -1$. Then*

$$-\beta(X+Y)^- \in \partial f_{\mathcal{P}^\circ}(X).$$

(Due to Lemma 5.6, such a Y exists.)

Proof. Let $X \in L^2_{\mathcal{F}}(\mathcal{E})$. In view of Lemma 5.5, it suffices to show that $Z^* = -\beta(X+Y)^-$ maximizes (42).

Step 1. Due to $f(X+Y) = f(X)+Y$ for all $Y \in L^0(\mathcal{F})$ an element $Z^* \in \mathcal{P}^\circ$ maximizes

$$\operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] \right) \tag{44}$$

if and only if it maximizes

$$\operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} \left(E[Z(X+Y) \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] \right).$$

Thus, we can assume that $E[-\beta X^- \mid \mathcal{F}] = -1$ since else we could replace X by $X+Y$ for the unique $Y \in L^0(\mathcal{F})$ with $E[-\beta(X+Y)^- \mid \mathcal{F}] = -1$.

Step 2. For all $Z \in \mathcal{P}^\circ$

$$E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] = E\left[ZX - \frac{1}{2\beta} Z^2 \mid \mathcal{F}\right] + \frac{1}{2\beta}.$$

Hence, $Z^* \in \mathcal{P}^\circ$ maximizes (44) if and only if it maximizes

$$\operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E\left[ZX - \frac{1}{2\beta} Z^2 \mid \mathcal{F}\right].$$

For $Z^* \in \mathcal{P}^\circ$ the following statements are equivalent:

(i)
$$E\left[Z^* X - \frac{1}{2\beta} Z^{*2} \mid \mathcal{F}\right] = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E\left[ZX - \frac{1}{2\beta} Z^2 \mid \mathcal{F}\right].$$

(ii) For all $Z \in \mathcal{P}^\circ$ and $\varepsilon \in [0,1]$,

$$E\left[Z^* X - \frac{1}{2\beta} Z^{*2} \mid \mathcal{F}\right] \geq E\left[Z_\varepsilon X - \frac{1}{2\beta} Z_\varepsilon^2 \mid \mathcal{F}\right],$$

where $Z_\varepsilon = \varepsilon Z + (1-\varepsilon) Z^*$. (Note that $Z_\varepsilon \in \mathcal{P}^\circ$ for all $Z \in \mathcal{P}^\circ$.)

(iii) For all $Z \in \mathcal{P}^\circ$,

$$\frac{d}{d\varepsilon} E\left[Z_\varepsilon X - \frac{1}{2\beta} Z_\varepsilon^2 \mid \mathcal{F}\right] |_{\varepsilon=0} \leq 0.$$

Indeed, for all $Z \in \mathcal{P}^\circ$ and $\varepsilon \in [0,1]$

$$E\left[Z_\varepsilon X - \frac{1}{2\beta} Z_\varepsilon^2 \mid \mathcal{F}\right] = \varepsilon Y_1 - \frac{\varepsilon^2}{2\beta} E[(Z - Z^*)^2 \mid \mathcal{F}] + Y_2$$

for some $Y_1 = Y_1(Z, Z^*), Y_2 = Y_2(Z, Z^*) \in F$. In particular, $\varepsilon \mapsto \varepsilon Y_1 - \frac{\varepsilon^2}{2\beta} E[(Z - Z^*)^2 \mid \mathcal{F}] + Y_2$ is point wise concave on $[0,1]$ and hence (iii) implies (ii).

(iv) For all $Z \in \mathcal{P}^\circ$, $E\left[(Z - Z^*)\left(X - \frac{1}{\beta} Z^*\right) \mid \mathcal{F}\right] \leq 0$.

Hence, in view of (iv) it follows that $Z^* = -\beta X^- \in \mathcal{P}^\circ$ maximizes (44). □

Example 5.8. *Again we employ the results of Section 5.3 in [FK07] to derive the above results for the specific case of $\mathcal{F} = \sigma(A_n)$, cf. Example 5.4. We identify the conditional mean variance, this time simply denoted by f, with its corresponding sequence of static mean variances $f = (f_1, f_2, f_3, \ldots)$.*

According to the above results, the greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{\mathcal{P}^\circ} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ identified with $f_{\mathcal{P}^\circ} = (f_{1,\mathcal{P}^\circ}, f_{2,\mathcal{P}^\circ}, f_{3,\mathcal{P}^\circ}, \ldots)$, where $f_{n,\mathcal{P}^\circ} : L^2(\mathcal{E} \cap A_n) \to \mathbb{R}$ is given by

$$f_{n,\mathcal{P}^\circ}(X) = \sup_{Z \in L^2(\mathcal{E} \cap A_n), Z \leq 0, E_{P_n}[Z] = -1} \left(E_{P_n}[ZX] - \frac{1}{2\beta} E_{P_n}[(1+Z)^2] \right),$$

for all $n \in \mathbb{N}$.

Alternatively, due to Section 5.3 in [FK07] the greatest monotone closed convex function majorized by f_n is given by the static monotone mean variance $f_{n,\mathcal{P}^\circ}$ for each $n \in \mathbb{N}$. Consequently, the greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by $f = (f_1, f_2, f_3, \ldots)$ must be $f_{\mathcal{P}^\circ} = (f_{1,\mathcal{P}^\circ}, f_{2,\mathcal{P}^\circ}, f_{3,\mathcal{P}^\circ}, \ldots)$.

A Proof of Theorem 2.4

In this appendix we provide a prove of Zowe's convex duality result in the form of Theorem 2.4. The setup and notation are as in Section 2. We first present a topological lemma.

Lemma A.1. *There exists a base of neighborhoods V of $0 \in L^k(\mathcal{G})$ such that*

$$V = (V + L^k_+(\mathcal{G})) \cap (V - L^k_+(\mathcal{G})), \tag{45}$$

where $L^k_+(\mathcal{G}) = \{X \in L^k(\mathcal{G}) \mid X \geq 0\}$, $k \in [1, +\infty]$ and $\mathcal{G} \subset \mathcal{E}$ denotes a generic sub σ–algebra of $\mathcal{E}$.

Proof. For each $n \in \mathbb{N}$ we denote by $B_{1/n}$ the ball of radius $1/n$ centered at $0 \in L^k(\mathcal{G})$. The collection $(B_{1/n})$ is the canonical base of neighborhoods in $L^k(\mathcal{G})$. We claim that $V_{1/n} = (B_{1/n} + L^k_+(\mathcal{G})) \cap (B_{1/n} - L^k_+(\mathcal{G}))$, $n \in \mathbb{N}$, defines a neighborhood base as required. Indeed, each $V_{1/n}$ satisfies (45) by construction. Further, $B_{1/n} \subset V_{1/n}$ and $V_{1/(2n)} \subset B_{1/n}$ for each $n \in \mathbb{N}$ implies that $(V_{1/n})$ is a base of neighborhoods. $\square$

The epigraph epi$f$ of a function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is understood as $\{(X, Y) \in L^p(\mathcal{E}) \times L^r(\mathcal{F}) \mid f(X) \leq Y\}$. The next lemma proves the first assertion of Theorem 2.4.

Lemma A.2. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a convex function. If f is continuous at $X_0 \in L^p(\mathcal{E})$ then f has a subgradient at X_0.*

Proof. The set

$$A = \mathrm{epi}f - \{(X_0, Y) \in \{X_0\} \times L^r(\mathcal{F}) \mid Y \leq f(X_0)\}$$

is nonempty and convex. Thus,

$$B = \bigcup_{\lambda \in [0,+\infty)} \lambda A$$

is a convex cone in $L^p(\mathcal{E}) \times L^r(\mathcal{F})$, that is $B + B \subset B$ and $\lambda B \subset B$ for all $\lambda \in [0, +\infty)$. Using B we will construct a sublinear mapping $p : L^p(\mathcal{E}) \to L^r(\mathcal{F})$, that is, p is subadditive $p(X_1 + X_2) \leq p(X_1) + p(X_2)$ and positively homogeneous $p(\lambda X_1) = \lambda p(X_1)$ for all $X_1, X_2 \in L^p(\mathcal{E})$ and $\lambda \in [0, +\infty)$. To this end, we define

$$S_X = \{Y \in L^r(\mathcal{F}) \mid (X, Y) \in B\},$$

for all $X \in L^p(\mathcal{E})$. We will show that S_X is nonempty and bounded from below for all $X \in L^p(\mathcal{E})$.

Since B is a convex cone we observe first that

$$S_{X_1} + S_{X_2} \subset S_{X_1+X_2}, \text{ for all } X_1, X_2 \in L^p(\mathcal{E}). \tag{46}$$

For $X \in L^p(\mathcal{E})$ we have

$$(X, f(X_0 + X) - f(X_0)) = (X_0 + X, f(X_0 + X)) - (X_0, f(X_0)) \in A,$$

and hence $(X, f(X_0 + X) - f(X_0)) \in B$. Thus,

$$S_X \neq \emptyset \text{ for all } X \in L^p(\mathcal{E}). \tag{47}$$

Let $(0, Y) \in B$, $Y \neq 0$. Then $(0, Y) = \lambda((X_1, Y_1) - (X_2, Y_2))$ for some $\lambda \in (0, +\infty)$, $X_1 = X_2 = X_0$ and $Y_1 \geq f(X_0) \geq Y_2$. Thus, $Y = \lambda(Y_1 - Y_2) \geq 0$, and hence

$$S_0 \subset L^r_+(\mathcal{F}). \tag{48}$$

For $X \in L^p(\mathcal{E})$ take $Y \in S_{-X}$ which is possible due to (47). From (46) and (48) we derive for all $Z \in S_X$

$$Z + Y \in S_X + S_{-X} \subset S_0 \subset L^r_+(\mathcal{F}).$$

Hence $-Y$ is a lower bound for S_X. Since $L^r(\mathcal{F})$ is order complete the mapping $p : L^p(\mathcal{E}) \to L^r(\mathcal{F})$,

$$p(X) = \text{ess.inf}\{Y \mid Y \in S_X\}$$

is well defined. Next, we show that p is sublinear.

For $\lambda \in (0, +\infty)$ we have $\lambda B = B$, and hence $\lambda p(X) = \text{ess.inf}\{\lambda Y \mid (X, Y) \in B\} = \text{ess.inf}\{\lambda Y \mid (\lambda X, \lambda Y) \in B\} = p(\lambda X)$. Since $p(0) = 0$ it follows that p is positively homogeneous. Further, from (46) we derive for all $X_1, X_2 \in L^p(\mathcal{E})$

$$p(X_1 + X_2) \leq Y_1 + Y_2, \text{ for all } Y_1 \in S_{X_1}, Y_2 \in S_{X_2}.$$

Thus, $p(X_1+X_2) \leq p(X_1)+p(X_2)$. Hence, p is subadditive and in turn sublinear. By the Hahn–Banach extension theorem in the form of Theorem 8.30 in [AB06] there exists a linear mapping $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ such that $\mu X \leq p(X)$ for all $X \in L^p(\mathcal{E})$. Since $f(X) - f(X_0) \in S_{X-X_0}$ for all $X \in L^p(\mathcal{E})$ we have

$$\mu(X - X_0) \leq p(X - X_0) \leq f(X) - f(X_0). \tag{49}$$

for all $X \in L^p(\mathcal{E})$. Thus, μ is a subgradient of f at X_0 if we can show that μ is continuous.

To this end, let V be a neighborhood of $0 \in L^r(\mathcal{F})$. We can assume that $V = -V$ and, due to Lemma A.1, $V = (V + L^r_+(\mathcal{F})) \cap (V - L^r_+(\mathcal{F}))$. Since f is continuous at X_0 there exists a symmetric neighborhood W ($W = -W$) of $0 \in L^p(\mathcal{E})$ such that

$$f(X_0 + W) \subset f(X_0) + V.$$

Hence, $f(X_0 + W) - f(X_0) \subset V$ and therefore

$$f(X_0 + X) - f(X_0) \in V \text{ for all } X \in W.$$

From (49) we find for all $X \in L^p(\mathcal{E})$ that $\mu X = \mu(X_0 + X - X_0) \leq f(X_0 + X) - f(X_0)$. Hence for all $X \in W = -W$

$$\mu X \in f(X_0 + X) - f(X_0) - L^r_+(\mathcal{F}) \subset V - L^r_+(\mathcal{F})$$

and

$$\mu X \in -(f(X_0 - X) - f(X_0) - L^r_+(\mathcal{F})) \subset -V + L^r_+(\mathcal{F}) = V + L^r_+(\mathcal{F}).$$

We conclude that $\mu(W) \subset (V + L^r_+(\mathcal{F})) \cap (V - L^r_+(\mathcal{F})) = V$ and continuity of μ follows at $0 \in L^p(\mathcal{E})$. Linearity of μ yields continuity on all of $L^p(\mathcal{E})$. $\square$

The second assertion of Theorem 2.4 can be proved as follows. We let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a convex function which is continuous at $X_0 \in L^p(\mathcal{E})$. We define $\mathrm{dom} f^{**} = \{X \in L^p(\mathcal{E}) \mid f^{**}(X) \in L^r(\mathcal{F})\}$. Lemma A.2 together with (3) yields $\mathrm{dom} f^* \neq \emptyset$ and we get

$$\mu X_0 - f^*(\mu) \leq \mu X_0 - (\mu X_0 - f(X_0)) = f(X_0), \text{ for all } \mu \in \mathrm{dom} f^*.$$

Hence, $X_0 \in \mathrm{dom} f^{**}$ and $f^{**}(X_0) \leq f(X_0)$. The reverse inequality follows from the observation that again Lemma A.2 together with (3) yields the existence of μ_0 such that $f(X_0) = \mu_0 X_0 - f^*(\mu_0)$ which concludes the proof.

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