Risk Preferences and their Robust Representation

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To address the plurality of interpretations of the subjective notion of risk, we describe it by means of a risk order and concentrate on the context invariant features of diversification and monotonicity. Our main results are uniquely characterized robust representations of lower semicontinuous risk orders on vector spaces and convex sets. This representation covers most instruments related to risk and allow for a differentiated interpretation depending on the underlying context which is illustrated in different settings: For random variables, risk perception can be interpreted as *model risk*, and we compute among others the robust representation of the economic index of riskiness. For lotteries, risk perception can be viewed as *distributional risk* and we study the "Value at Risk". For consumption patterns, which excerpt an intertemporality dimension in risk perception, we provide an interpretation in terms of *discounting risk* and discuss some examples.

Keywords: Risk Perception, Risk Measures, Robustness

Introduction

Risk is now a colloquial and widely used term. Nevertheless, its emergence in history is relatively recent[1]. Actually, "the late apparition in history of circumstances indicated by means of the new term 'risk' is probably due to the fact that it accommodates a plurality of distinctions within one concept, thus constituting the unity of this plurality", Luhmann [42, Page 16]. Many scientific areas ranging from economics and finance to sociology or medicine have now laid claim to this concept with their own instruments, language, and objectives focusing on different types of risk such that, even today, when it comes to specify this notion, ways are parting and no real consensus emerges. Yet, at the beginning of the twentieth century, discussing in an economical context his famous distinction between *measurable uncertainty*, where an objective probability can be assigned to uncertain outcomes, and *unmeasurable uncertainty*, where no a priori probability can be provided, Knight proposed to identify risk with measurable uncertainty[2]. This Knightian distinction between unmeasurable uncertainty and his notion of risk identified with measurable uncertainty has strongly influenced modern economic thought; in particular the subsequent developments of decision theory where properties reflecting a *normative* view of rationality in the expression of preferences in face of uncertainty are studied.

von Neumann and Morgenstern [53] initiated this approach with their work on preferences over *lotteries*, which under some conditions admit a numerical representation in terms of an expected utility. An important milestone though, beyond this paradigm of expected utility for which many extension were given, is the axiomatic approach of Gilboa and Schmeidler [34]. Motivated from a decision theoretical viewpoint, they obtain a representation in terms of the worst expected utility of a random variable evaluated with respect to different probability models. Such a *robustification* of the expected utility has further been extended by Klibanoff, Marinacci, and Mukerji [39], Maccheroni, Marinacci, and Rustichini [43], Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [16]. From another viewpoint, motivated by the need of regulatory agencies for a method of specifying capital requirements for financial institutions, Artzner, Delbaen, Eber, and Heath [4] introduced an axiomatic framework to describe the risk of financial products. This led to the concept of *coherent cash additive risk measures* which are characterized by representations taking the worst of expected losses of a random variable evaluated with respect to different probability models. This concept has been further extended to the concept of *convex cash additive risk measures* introduced independently by Föllmer and Schied [29], Frittelli and Rosazza Gianin [33] and Heath [36].

In view of these normative approaches, let us come back to the Knightian identification between risk and measurable uncertainty. Actually, the developments in the theory of preferences and cash additive risk measures suggest to separate these two notions. Indeed, the well founded use of the term risk in the theory of cash additive risk measures does not match Knight's notion of risk. By way of their representation, cash additive risk measures turn out to address risk in the sense of "unmeasurable uncertainty", since different probability models, rather than a single one, are taken into account just as in Gilboa and Schmeidler's robustification of expected utility. Throughout this work, in line with Knight [40] and Keynes [38, P. 213–214], *uncertainty* describes merely the fact that a situation might have more than one possible outcome, for instance, situations of prospective nature. *Unmeasurable uncertainty* corresponds to situations where we simply can not know, and *measurable uncertainty* applies to situations where some quantification can take place. Such a quantification might for instance involve a set of probability models as illustrated in

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[1]The term "risicum" already appears in the Middle Ages in highly specific contexts, but Luhmann [41] traces its wider use and the diversification of its meaning to the early Renaissance.

[2]"To preserve the distinction [...] between the measurable uncertainty and an unmeasurable one we may use the term 'risk' to designate the former and the term 'uncertainty' for the latter" [40, Part III, Chapter VIII, Paragraph 1].

Section 3.1 or a set of discounting functions, as in Section 3.3. Now, even if risk is intimately related to uncertainty, in contrast to Knight, we do not identify it with measurable uncertainty. Indeed, while uncertainty, measurable or not, is an inherent quality of a situation, risk is not. Risk is definitively a *subjective* notion since "causal terms and terms like risk or danger are not indications of ontological facts about which one can have only true or false opinions. [...] Risk evaluation is not simply a problem of avoiding an error. The question rather is: who uses which frame to guide his observations; and then, who observes others handle causal distinctions and how they discriminate external and internal attribution depending upon whether they themselves or others make the decisions" [41, Page 6]. Luhmann underlines thereby also that risk is rather a matter of *perception* which clearly depends on one's perspective and the surrounding context.

These different considerations on uncertainty and risk are our primary motivations to approach the risk perception of uncertainty in a sense which firstly reflects its subjective nature and secondly allows for a context and perspective dependent interpretation. The emphasis on the subjective dimension is the reason to consider the risk perception of elements subject to uncertainty by means of a preference order $\preccurlyeq$ where the relation $x \preccurlyeq y$ means "the element x is perceived to be less risky than the element y". To characterize this preference order as a perception of risk while keeping track of the context dependency, we concentrate on invariant key features commonly related to risk. These are expressed by the normative statements "diversification should not increase the risk" and "the better for sure, the less risky". A preference order reflecting such properties will be called a *risk order*. Beyond the fact that there is a broad consensus that diversification and monotonicity capture crucial features of risk perception, they leave full latitude in which setting they are considered and how they might be specified. Since monotonicity is formulated by means of an arbitrary preorder[3] and diversification by an arbitrary convex structure, their specification induces different notions and interpretations.

The first advantage of our approach is that it covers most of the instruments related in one way or another to risk: von Neumann and Morgenstern's expected utilities, Markowitz [45]'s mean variance, Sharpe [52]'s ratio, "Value at Risk", Gilboa and Schmeidler [34]'s robustification of expected utilities, monetary risk measures, Aumann and Serrano [5]'s economic index of riskiness, or Cherny and Madan [21]'performance measures, to name but a few. As for the second advantage, our approach provides the key instruments for the interpretation of risk perception under different perspectives by means of our main result, a *robust representation* of risk orders on general convex sets of the form

$$\rho\left(x\right) = \sup_{x^*} R\left(x^*, \langle x^*, -x\rangle\right). \tag{0.1}$$

In this representation, the *risk measure* ρ is a numerical representation of the considered risk order and R is a *uniquely* characterized *maximal risk function*. As for the general meaning of such a representation, the risk of the "losses" of an uncertain element x is estimated under a evaluation x^* by means of the operation $\langle x^*, -x\rangle$. However, since a risk perceiver is not sure which evaluation is adequate to estimate the "losses", other evaluations are considered weighted according to their plausibility by means of the maximal risk function R. Finally, a precautious approach is adopted by taking the maximum of those estimations justifying henceforth the terminology "robust". This schema is generic as for the expression of risk perception; as for the key instruments for the interpretation, they are given by the nature of the uncertain elements x and the corresponding set of evaluations x^*. Let us briefly illustrate this robust representation for different settings for which diversification and monotonicity correspond to radically different notions and interpretations. On the level of random variables X with the standard notion of

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[3]For instance the relation "... greater or equal than ... almost surely" in the setting of random variables.

monotonicity "greater than almost surely", the robust representation reduces to

$$\rho\left(X\right) = \sup_Q R\left(Q, E_Q\left[-X\right]\right) \tag{0.2}$$

where the evaluations Q are probability measures allowing to view risk perception as *model risk*. On the level of lotteries μ where diversification corresponds to some additional randomization, and the monotonicity is specified by the first stochastic order, the risk perception is characterized by a robust representation

$$\rho\left(\mu\right) = \sup_l R\left(l, \int l\left(-x\right)\mu\left(dx\right)\right) \tag{0.3}$$

where the evaluations l are increasing loss functions. Risk perception may here be interpreted as *distributional risk*. On the level of consumption patterns c, the intertemporal dimension in the risk perception has a representation of the form

$$\rho\left(c\right) = \sup_\beta R\left(\beta, -\int_0^1 \beta_s dc_s\right) \tag{0.4}$$

where the evaluations β are some discounting functions yielding a risk perception in terms of *discounting risk*. In conclusion, our approach to risk perception is in accordance with Luhmann's quotation in the first paragraph, since it allows for a plurality of interpretations within one framework depending on the context and one's perspective.

We next discuss and compare related results to our robust representation (0.1). Penot and Volle [49] have obtained some duality results of the form (0.1) for quasiconvex functionals. However, they do not address the issues of monotonicity and uniqueness, the latter being crucial for comparative statics in terms of the risk function R. Recently, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [16, 14, 15] proved the uniqueness of the representation (0.1) under the denomination of a *complete quasiconvex duality* result. We wish to point out that their results and the signification of quasiconvexity in preferences theory were a crucial source of inspiration for the present work. While their results address a complete quasiconvex duality for evenly or upper semicontinuous functions in the setting of M-spaces[4] or $\mathbb{L}^p$ spaces, our main result is a complete quasiconvex duality for lower semicontinuous functions on convex subsets of general locally convex vector spaces. One of the advantages of the technical assumption of lower semicontinuity is that it is often a consequence of the monotonicity, see Remark 2.8. Our approach allows us in particular to consider other types of risk perceptions besides the setting of bounded random variables such as lotteries or consumption patterns as studied in Section 3. In the specific setting of lotteries, a related representation has independently been achieved by Cerreia-Vioglio [13]. There, convex preferences on the set of simple lotteries over a general set are considered. He furthermore investigates convex preferences over sets of so called menus, yielding a maxmin representation. In contrast, in Section 3.2, we study lower semicontinuous risk orders over general lotteries with compact support on the real line, which are monotone with respect to the first stochastic order.

The subsequent work is structured as follows. In the first section, we introduce the concept of risk orders on convex sets of uncertain elements. We further introduce *risk measures* which are quasiconvex monotone functions playing the role of numerical representation of risk orders, and define *risk acceptance families*. The main result of the first section is Theorem 1.7 which clarifies the one-to-one correspondence

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[4]A typical example of an M-space is the set of bounded random variables.

between risk orders, risk measures, and risk acceptance families. In Subsection 1.1 and 1.2, we study additional properties of risk measures such as convexity, positive homogeneity, scaling invariance, affinity, cash additivity or cash subadditivity. In the second section, we give a dual representation of lower semicontinuous risk orders in the setting of locally convex topological vector spaces. The main Theorem 2.6, shows that any lower semicontinuous risk measure ρ of a lower semicontinuous risk order $\preccurlyeq$ admits a robust representation of the form[5] (0.1). Furthermore, we characterize the class of maximal[6] risk functions R for which the uniqueness in this representation holds. If the monotonicity preorder satisfies a regularity condition, Theorem 2.7 states that the supremum in (0.1) can actually be taken over a smaller set of normalized evaluations[7]. The delicate question of the existence and uniqueness of a robust representation on a convex subset of a vector space is addressed in Subsection 2.1. Indeed, we provide an example where the uniqueness in the class of maximal risk functions is no longer ensured when considering risk measures on a convex subset. Under the assumption of continuous extensibility, which automatically holds for many convex sets, we provide in Theorem 2.19 a characterization of a smaller class of maximal risk functions for which a complete quasiconvex duality result holds on convex sets. This result covers the important settings of lotteries, or consumption patterns. In the third section we study how our robust representation applies in several illustrative settings. Beside the interest in terms of a differentiated interpretation of risk perception, it also illustrates how the risk acceptance family plays a central role in explicit computations of maximal risk functions. In the setting of random variables, we discuss how the Fatou property provides a robust representation of the form (0.2) in terms of probability measures. As an example, we compute robust representations of various certainty equivalents, and the economic index of riskiness. In the setting of lotteries, we derive a unique robust representation of the form (0.3) where the set of maximal risk functions is remarkably identical with the one of the respective vector space. A discussion and a robust representation of the "Value at Risk" illustrates then the importance of the setting since it is quasiconvex on the level of lotteries but not on the level of random variables. Diversification on the level of consumption patterns excerpts another interesting dimension of risk perception, namely the one related to intertemporal relations. We derive a robust representation on this convex set of the form (0.4) which we illustrate by an intertemporal risk measure inspired by Hindy, Huang, and Kreps [37]. Finally, Theorem 3.10 in Subsection 3.4 illustrate the interplay between model risk and distributional risk in the setting of Anscombe and Aumann [3]. The appendix collects standard mathematical concepts and all the technical proofs.

1 Risk Orders, Risk Measures and Risk Acceptance Families

Throughout, we study the risk of elements x in some nonempty space $\mathcal{X}$. The risk perception is specified by a *total preorder*[8] on $\mathcal{X}$ denoted by $\preccurlyeq$. As usual, the notations $\prec := \{\preccurlyeq\ \&\ \not\succcurlyeq\}$ and $\sim := \{\preccurlyeq\ \&\ \succcurlyeq\}$ respectively correspond to the antisymmetric and equivalence relation. A *numerical representation* of a total preorder $\preccurlyeq$ is a mapping $F : \mathcal{X} \to [-\infty, +\infty]$, such that

$$x \preccurlyeq y \quad \text{if and only if} \quad F(x) \leq F(y) \tag{1.1}$$

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[5]The set of evaluations over which the supremum in (0.1) is taken is the polar cone determined by the preorder used to define the monotonicity of the risk order.

[6]The term "maximal" risk function is justified in a pointwise sense by Proposition 2.9.

[7]In the case of bounded random variables with monotonicity preorder "greater than almost surely", the normalized subset corresponds to the set of probability measures.

[8]A *preorder* is a binary relation $\preccurlyeq$ on $\mathcal{X}$, which is reflexive and transitive. A binary relation $\preccurlyeq$ is reflexive if $x \preccurlyeq x$ for all $x \in \mathcal{X}$, and transitive if $x \preccurlyeq y$ and $y \preccurlyeq z$ implies $x \preccurlyeq z$. A *total preorder* is a preorder which in addition is complete, that is, $x \preccurlyeq y$ or $y \preccurlyeq x$ for all $x, y \in \mathcal{X}$. Note that a complete binary relation is reflexive.

for all $x, y \in \mathcal{X}$. It is well-known, see Debreu [24, 25], that a total preorder $\preccurlyeq$ admits a numerical representation if and only if it is *separable*[9]. It is also straightforward to check that a numerical representation of $\preccurlyeq$ is unique up to *increasing transformation*, that is, for any two numerical representations $F, \hat{F}$ of $\preccurlyeq$ there exists an increasing function $h : Im(F) \to Im(\hat{F})$ such that $\hat{F} = h \circ F$.

Our aim is to characterize those total preorders which deserve the denomination "risk". As evoked in the introduction, the main properties related to risk perception are the diversification and some form of monotonicity. In order to diversify risky elements, we need to express convex combinations, thus, $\mathcal{X}$ is from now on a convex subset[10] of a vector space $\mathcal{V}$. As for the monotonicity, we need a relation specifying that some elements are "better for sure" than others, which is expressed by a preorder $\trianglerighteq$ on $\mathcal{X}$. Throughout, we assume that $\trianglerighteq$ is even a *vector preorder*[11]. A total preorder $\preccurlyeq$ which reflects the diversification and monotonicity properties is called a risk order.

Definition 1.1 (Risk Order). A total preorder $\preccurlyeq$ on $\mathcal{X}$ is a *risk order* if it is

- *quasiconvex:* $\lambda x + (1 - \lambda) y \preccurlyeq y$ for any $\lambda \in \,]0, 1[$ whenever $x \preccurlyeq y$,

- *monotone:* $x \preccurlyeq y$ whenever $x \trianglerighteq y$.

Here, the relation $x \preccurlyeq y$ means "*x is less risky than y*". The quasiconvexity axiom reflects exactly that the diversification between two alternatives keeps the risk below the worse one. The monotonicity axiom states that the risk order is compatible with the preorder $\trianglerighteq$. In the following, $\mathcal{L}(x) = \{y \in \mathcal{X} : y \preccurlyeq x\}$ consists of those elements which are less risky than $x \in \mathcal{X}$. Note that a total preorder $\preccurlyeq$ is quasiconvex exactly when $\mathcal{L}(x)$ is convex for all $x \in \mathcal{X}$.

Remark 1.2. Note that the monotonicity can be ruled out if the vector preorder $\trianglerighteq$ is trivial, that is, the relation $x \trianglerighteq y$ holds if and only if $x = y$. In that case, we say that the risk order is monotone with respect to the trivial preorder. ♦

The abstractness of the setting agrees with our declared intention to concentrate solely on the properties characterizing the risk perception as such. This allows us to appreciate and interpret it under different lights depending on the underlying context. We precise this thereafter with several illustrative settings which will be studied in Section 3.

- **Random Variables:** In finance, risky positions—equities, credits, derivative products, insurance contracts, portfolios, etc.—are commonly random variables defined on some state space Ω. Capital letters $X, Y, \ldots$ are usually used instead of $x, y, \ldots$ to refer to those risky positions. A possible choice for $\mathcal{X}$ is the vector space $\mathbb{L}^\infty := \mathbb{L}^\infty(\Omega, \mathscr{F}, P)$[12], where P is a reference probability measure defined on a σ-algebra of possible scenarios $\mathscr{F}$. The diversification is expressed by the state-wise convex combination $\lambda X(\omega) + (1 - \lambda) Y(\omega)$ for P-almost all ω, and the canonical preorder is given by the relation "greater than P-almost surely".

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[9] A total preorder $\preccurlyeq$ is *separable* if there exists a countable set $\mathcal{Z} \subseteq \mathcal{X}$ such that for any $x, y \in \mathcal{X}$ with $x \prec y$ there is $z \in \mathcal{Z}$ for which $x \preccurlyeq z \preccurlyeq y$.

[10] The framework of a mixture space could have been considered as well, at least in the first section. However, up to two reasonable additional conditions (non triviality, and a weak form of associativity), mixture spaces can be embedded as a convex subset of a vector space, see [46].

[11] A *vector preorder* $\trianglerighteq$ is the restriction to $\mathcal{X}$ of a preorder $\trianglerighteq$ defined on the vector space $\mathcal{V} \supset \mathcal{X}$ such that $x \trianglerighteq y$ implies $x + z \trianglerighteq y + z$ for any $z \in \mathcal{V}$ and $\lambda x \trianglerighteq \lambda y$ for any $\lambda \geq 0$. A vector preorder is specified by the convex cone $\mathcal{K} := \{x \in \mathcal{X} : x \trianglerighteq 0\}$, for which $x \trianglerighteq y$ exactly when $x - y \in \mathcal{K}$.

[12] $\mathbb{L}^\infty(\Omega, \mathscr{F}, P)$ denotes the vector space of all essentially bounded random variables, where random variables are identified when they coincide P-almost surely.

- **Lotteries:** Historically, probability distributions, referred to as lotteries, play an important role in decision theory. Here also, the tradition sees the use of the notation $\mu, \nu, \ldots$ instead of $x, y, \ldots$ We will consider the set $\mathcal{M}_{1,c}$ of lotteries with compact support on the real line. Here, a convex combination $\lambda\mu + (1 - \lambda)\nu$ can be interpreted as some additional randomization, since it corresponds to the sampling of either the lottery μ or ν depending on the outcome of a binary lottery with probability λ or $(1 - \lambda)$. As a convex set, $\mathcal{M}_{1,c}$ spans the vector space ca_c of *bounded signed measures* with compact support. Different orders might be considered on $\mathcal{M}_{1,c}$ such as the *first* or *second stochastic order* defined by $\mu \trianglerighteq \nu$ if $\int f\, d\mu \geq \int f\, d\nu$ for all continuous nondecreasing, respectively concave continuous nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$.

- **Consumption Patterns:** They are particularly adequate to excerpt the intertemporal dimension in the perception of successive consumption of a commodity, in particular the substitution effects. To take into account gulps along continuity, these consumption patterns in time are modelled by nondecreasing right-continuous paths $c : [0,1] \to [0, +\infty[$, where the value c_t represents the cumulative amount of consumption up to time $t \in [0,1]$. Diversification expressed by means of the time-wise convex combination $\lambda c_t^{(1)} + (1 - \lambda)\, c_t^{(2)}$ for all $0 \leq t \leq 1$. The set of consumption patterns denoted by $\mathcal{CS} := \mathcal{CS}([0,1])$ is a convex cone. A possible preorder is defined by $c^{(1)} \trianglerighteq c^{(2)}$ exactly when $c^{(1)} - c^{(2)}$ belongs to $\mathcal{CS}$.

- **Stochastic Kernels:** They can be seen as lotteries which are additionally subject to model uncertainty. These state dependant lotteries denoted by $\tilde{\mu} := \tilde{\mu}(\omega, dx)$ unify somehow lotteries and random variables in one object and can be used to illustrate the interplay between model uncertainty and distributional uncertainty in the risk perception[13]. Mathematically, the set of stochastic kernels $\mathcal{SK}$ consists of all measurable mappings[14] $\tilde{\mu} : \Omega \to \mathcal{M}_{1,c}$, where $(\Omega, \mathscr{F}, P)$ is a probability space. Convex combinations are ω-wise randomizations between state dependant lotteries, $\lambda\tilde{\mu}(\omega, dx) + (1 - \lambda)\tilde{\nu}(\omega, dx)$. As for the preorder we consider the P-almost sure second stochastic order, that is, $\tilde{\mu} \trianglerighteq \tilde{\nu}$ if the lottery $\tilde{\nu}(\omega, \cdot)$ dominates in the second stochastic order the lottery $\tilde{\nu}(\omega, \cdot)$ for P-almost all $\omega \in \Omega$.

Remark 1.3. The notion of diversification strongly depends on the underlying setting. This can be seen in the difference between the settings of random variables and lotteries. Indeed, the randomization $\lambda P_X + (1 - \lambda) P_Y$ of lotteries P_X and P_Y corresponding to the laws of random variables X and Y under a probability measure P generally differs from the lottery $P_{\lambda X + (1-\lambda)Y}$ corresponding to the law of the state-wise convex combination $\lambda X + (1 - \lambda) Y$. This has crucial consequences for the interpretation in terms of risk perception as illustrated in the following example. Suppose that a lender faces a choice between two similar but independent loans each with a default probability of one percent. Assume also that the risk perception of this lender is focused on the probability of loosing money. The lender estimates therefore a mere convex combination of half a loan and half the other as more risky since the probability of loosing money increases to almost two percents, even if the loss size is reduced. The perspective of its perception is however explicitly in a distributional sense rather than in terms of values. From this point of view, diversification is understood on the level of lotteries, where such a convex combination corresponds to a coin toss before choosing one loan or the other. By doing so, the probability of loosing money remains at one percent. ♦

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[13]They were first used in economic theory by Anscombe and Aumann [3] and further by Gilboa and Schmeidler [34] for their maximin expected utilities with multiple priors. See also [30, 39, 43, 14] among others.

[14]$\mathscr{F}$-$\mathscr{P}(\mathcal{M}_{1,c})$-measurable, where $\mathscr{P}(\mathcal{M}_{1,c})$ is the σ-algebra generated by the mapping $\mu \mapsto \mu(A)$ for any Borel set $A \subseteq \mathbb{R}$.

Numerical representations of risk orders inherit the key properties of risk perception and thus are from now on called risk measures and generically denoted by ρ. In Theorem 1.7 a correspondence between risk measures and risk orders will be given which justifies the following definition.

Definition 1.4 (Risk Measure). A mapping $\rho : \mathcal{X} \to [-\infty, +\infty]$ is called a *risk measure* if it is

- **quasiconvex:** $\rho\left(\lambda x + (1 - \lambda)y\right) \leq \max\{\rho\left(x\right), \rho\left(y\right)\}$ for all $x, y \in \mathcal{X}$ and $\lambda \in \left]0, 1\right[$,

- **monotone:** $\rho\left(x\right) \leq \rho\left(y\right)$ whenever $x \trianglerighteq y$.

Example 1.5. The *certainty equivalent* of an expected loss can be considered on the level of lotteries,

$$\rho\left(\mu\right) := l^{-1}\left(\int l\left(-x\right)\mu\left(dx\right)\right), \quad \mu \in \mathcal{M}_{1,c}, \tag{1.2}$$

where $l : \mathbb{R} \to \mathbb{R}$ is a continuous increasing loss function. Since the increasing function l^{-1} is clearly quasiaffine[15] it follows that ρ is monotone with respect to the first stochastic order and

$$\rho\left(\lambda\mu + (1 - \lambda)\nu\right) = l^{-1}\left(\lambda\int l\left(-x\right)\mu\left(dx\right) + (1 - \lambda)\int l\left(-x\right)\nu\left(dx\right)\right) \leq \max\left\{\rho\left(\mu\right), \rho\left(\nu\right)\right\},$$

for all $\mu, \nu \in \mathcal{M}_{1,c}$ and $\lambda \in \left]0, 1\right[$ and so ρ is a risk measure.

On the level of random variables, the certainty equivalent of an expected loss is defined as

$$\hat{\rho}\left(X\right) := l^{-1}\left(E\left[l\left(-X\right)\right]\right), \quad X \in \mathbb{L}^{\infty}. \tag{1.3}$$

Even though $\hat{\rho}\left(X\right) = \rho\left(P_X\right)$, the diversification on the level of random variables is different and the loss function l has to be additionally convex to ensure that $\hat{\rho}$ is a risk measure, see [15] where a robust version of it with more than an loss function has also been studied. Explicit computation of the robust representation of this risk measure will be given in Section 3, Example 3.3. $\diamond$

Before stating the relation between risk orders and risk measures, we introduce another concept crucial for the further understanding of this work. Given a risk measure ρ, for any *risk level* $m \in \mathbb{R}$, we define the *risk acceptance set of level* m as the subset $\mathcal{A}_{\rho}^{m} \subseteq \mathcal{X}$ of those elements having a risk smaller than m, that is

$$\mathcal{A}_{\rho}^{m} = \left\{x \in \mathcal{X} : \rho\left(x\right) \leq m\right\}, \quad m \in \mathbb{R}. \tag{1.4}$$

We call $\mathcal{A}_{\rho} = \left(\mathcal{A}_{\rho}^{m}\right)_{m \in \mathbb{R}}$ the *risk acceptance family* associated to ρ. Here again, the risk acceptance family carries the specificities of the risk measure. In Theorem 1.7 we will state a one-to-one relation between risk measures and risk acceptance families which satisfy the following conditions.

Definition 1.6 (Risk Acceptance Family). An increasing[16] family $\mathcal{A} = \left(\mathcal{A}^{m}\right)_{m \in \mathbb{R}}$ of subsets $\mathcal{A}^{m} \subseteq \mathcal{X}$ is a *risk acceptance family* if it is

- **convex:** $\mathcal{A}^{m}$ is a convex subset of $\mathcal{X}$ for all $m \in \mathbb{R}$,

- **monotone:** $x \in \mathcal{A}^{m}$ and $y \trianglerighteq x$ implies $y \in \mathcal{A}^{m}$,

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[15]Any nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$ is automatically quasiaffine, a definition of which is given in Appendix A.
[16]That is, $\mathcal{A}^{m} \subseteq \mathcal{A}^{n}$ for any $m \leq n$.

• **right-continuous:** $\mathcal{A}^m = \bigcap_{n>m} \mathcal{A}^n$ for all $m \in \mathbb{R}$.

The risk acceptance family is not only a major instrument for the robust representation of risk measures in Section 2, it can also be used to describe further structural properties or to model specific economical features of risk, see Example 1.22.

The following Theorem states the one-to-one correspondence between risk orders, risk measures, and risk acceptance families.

Theorem 1.7. *Any numerical representation $\rho_{\preccurlyeq} : \mathcal{X} \to [-\infty, +\infty]$ of a risk order $\preccurlyeq$ is a risk measure. Conversely, any risk measure $\rho : \mathcal{X} \to [-\infty, +\infty]$ defines a risk order $\preccurlyeq_\rho$ through*

$$x \preccurlyeq_\rho y \quad \text{if and only if} \quad \rho(x) \le \rho(y). \tag{1.5}$$

Moreover, $\preccurlyeq_{\rho_{\preccurlyeq}} = \preccurlyeq$ and $\rho_{\preccurlyeq_\rho} = h \circ \rho$ for some increasing transformation h.

Furthermore, for any risk measure ρ, the family $\mathcal{A}_\rho$ given by

$$\mathcal{A}_\rho^m := \{x \in \mathcal{X} : \rho(x) \le m\}, \quad m \in \mathbb{R}, \tag{1.6}$$

is a risk acceptance family. Conversely, for any risk acceptance family $\mathcal{A}$, the functional $\rho_{\mathcal{A}}$ given by

$$\rho_{\mathcal{A}}(x) := \inf \{m \in \mathbb{R} : x \in \mathcal{A}^m\}, \quad x \in \mathcal{X}, \tag{1.7}$$

defines a risk measure. Moreover, $\rho_{\mathcal{A}_\rho} = \rho$ and $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$.

Proof, Appendix C.1.

The idea of expressing the numerical representation of a total preorder by means of an increasing family of acceptance sets as in (1.7) was recently used in other works: Cherny and Madan [21] characterize a class of *performance measures* built upon a specific family of acceptance sets[17] and Brown, De Giorgi, and Sim [10] represent a type of prospective preferences also by means of acceptance sets which are not necessarily convex.

Remark 1.8. While the convexity and monotonicity of a risk acceptance family reflect the key properties of risk perception, the right-continuity is needed to ensure the one-to-one correspondence. Indeed, in the proof of Theorem 1.7, it turns out that $\rho_{\mathcal{A}}$ is a risk measure even if the risk acceptance family $\mathcal{A}$ is not right-continuous. Nevertheless, it plays a crucial role for the relation[18] $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$. Be aware that the right-continuity condition for the risk acceptance family is not of topological nature. ♦

For notational convenience we mostly drop the reference indices and simply write $\mathcal{A}$, $\preccurlyeq$, ρ instead of $\mathcal{A}_\rho$, $\preccurlyeq_\rho$, $\rho_{\preccurlyeq}$ or $\rho_{\mathcal{A}}$, respectively. We now illustrate the previous theorem with the following two families of risk measures.

Example 1.9. Introduced by Föllmer and Schied [29], the *shortfall risk measure* is of additive nature and given by

$$\rho(X) := \inf \{s \in \mathbb{R} : E[l(-X-s)] \le c_0\}, \quad X \in \mathbb{L}^\infty, \tag{1.8}$$

where $E[l(-X)]$ is the expected loss of the position X according to a lower semicontinuous convex loss function $l : \mathbb{R} \to]-\infty, +\infty]$ increasing on its domain and such that $l(s_0) < +\infty$ for some $s_0 > 0$.

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[17]The risk acceptance family corresponds to the acceptance sets of a family of coherent monetary risk measures, see Section 1.2.

[18]Indeed, on $\mathcal{X} = \mathbb{R}$, consider the family $\mathcal{A}^m =]-m, +\infty[$ which is monotone and convex but fails to be right-continuous since $\mathcal{A}^m \neq [-m, +\infty[= \bigcap_{n>m}]-n, +\infty[$. Here, $\rho_{\mathcal{A}}(x) = -x$ and $\mathcal{A}_{\rho_{\mathcal{A}}}^m = [-m, +\infty[$ for all $m \in \mathbb{R}$, showing that $\mathcal{A} \neq \mathcal{A}_{\rho_{\mathcal{A}}}$.

This risk measure accounts for the minimal amount of money which added to the position X pulls its expected loss below a given threshold c_0 in the range of $l\left(\left]0,+\infty\right[\right)$. By the strict monotonicity and the lower semicontinuity of the loss function l holds

$$\mathcal{A}^m = \{X \in \mathbb{L}^\infty : \rho\left(X\right) \leq m\} = \{X \in \mathbb{L}^\infty : c_0 \geq E\left[l\left(-X - m\right)\right]\},$$

for any risk level $m \in \mathbb{R}$. Since $X \mapsto E\left[l\left(-X - m\right)\right]$ is convex and monotone, we deduce that $\mathcal{A}$ is a risk acceptance family and therefore, by means of Theorem 1.7, ρ is a risk measure. ◊

Example 1.10. First introduced by Aumann and Serrano [5] in the exponential case and extended to the logarithmic case by Foster and Hart [32], the *economic index of riskiness*, similar to the shortfall risk measure but of multiplicative nature, fits particularly well for returns. It can be generalized and interpreted as follows. We first define

$$\lambda\left(X\right) = \sup\{\lambda > 0 : E\left[l\left(-\lambda X\right)\right] \leq c_0\},$$

which represents the maximal exposure to a position $X \in \mathbb{L}^\infty$ provided that the expected loss remains below an acceptable level c_0 in the range of $l\left(\left]0+\infty\right[\right)$. Here, l is a loss function as in Example 1.9, which in addition fulfills the growth condition[19] $\lim_{x \to +\infty} l\left(x\right)/x = +\infty$. The economic index of riskiness is then defined[20] as

$$\rho(X) := \frac{1}{\lambda(X)}, \quad X \in \mathbb{L}^\infty.$$

Given a risk level[21] $m > 0$ holds

$$\mathcal{A}^m = \{X \in \mathbb{L}^\infty : \lambda\left(X\right) \geq 1/m\} = \{X \in \mathbb{L}^\infty : c_0 \geq E\left[l\left(-X/m\right)\right]\}.$$

Due to the convexity and the monotonicity of $X \mapsto E\left[l\left(-\lambda X\right)\right]$ it follows that $\mathcal{A}$ is convex and monotone and thus a risk acceptance family. Therefore, in view of Theorem 1.7, the economic index of riskiness is a risk measure. The loss functions in [5, 32] correspond to $l\left(s\right) = e^s - 1$ and $l\left(s\right) = -\ln\left(1 - s\right)$, respectively. A computation of the robust representation will be given in Section 3, Example 3.4. ◊

Remark 1.11. Theorem 1.7 ensures that as soon as of one of these objects—risk order, risk measure or risk acceptance family—is given, the other two are simultaneously precised. The notion of quasiconvexity and monotonicity are therefore global features, since any numerical representation of a risk order shares these properties and vice versa. In the following subsections, we will study additional properties of risk measures, such as convexity, affinity, or cash additivity amongst others. Unlike quasiconvexity and monotonicity, most of them do not hold for the entire class of numerical representations of the corresponding risk order and are in this sense no longer global. In the following, we call a risk order convex, affine, cash additive, etc., when there exists at least one numerical representation having this property. ♦

1.1 Further Structural Properties

As mentioned in Remark 1.11, the properties of convexity, positive homogeneity, or affinity are no longer global and are therefore defined on the level of risk measures.

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[19] Due to the monotonicity and convexity of l this growth condition insures the intuitive idea that expected losses are inflated more than gains since for any $X \in \mathbb{L}^\infty$ taking negative values on a set of positive probability, $E\left[l\left(-\lambda X\right)\right] \to +\infty$ for $\lambda \to +\infty$.

[20] With the usual conventions $1/0 = +\infty$ and $1/+\infty = 0$.

[21] Clearly, for any $m < 0$ holds $\mathcal{A}_\rho^m = \emptyset$, and $\mathcal{A}_\rho^0 = \mathbb{L}_+^\infty$ which are both convex.

Definition 1.12. A risk measure $\rho : \mathcal{X} \to [-\infty, +\infty]$ is

- **convex** if $\rho(\lambda x + (1 - \lambda) y) \le \lambda \rho(x) + (1 - \lambda) \rho(y)$ for all $x, y \in \mathcal{X}$ and $\lambda \in {]}0, 1{[}$.

- **positive homogeneous** if $\rho(\lambda x) = \lambda \rho(x)$ for all $x \in \mathcal{X}$ and $\lambda > 0$.

- **scaling invariant** if $\rho(\lambda x) = \rho(x)$ for all $x \in \mathcal{X}$ and $\lambda > 0$.

- **affine** if ρ is real-valued and $\rho(\lambda x + (1 - \lambda) y) = \lambda \rho(x) + (1 - \lambda) \rho(y)$ for all $x, y \in \mathcal{X}$ and $\lambda \in {]}0, 1{[}$.

The notions of positive homogeneity and scaling invariance require in addition that $\mathcal{X}$ is a convex cone. In line with Remark 1.11, we call a risk order $\preccurlyeq$ convex, positive homogeneous, etc. if it can be represented by a risk measure which has this property.

Proposition 1.13.   *(i) A risk measure ρ is convex if and only if the corresponding risk acceptance family $\mathcal{A}$ is level convex, that is, $\lambda \mathcal{A}^m + (1 - \lambda) \mathcal{A}^{m'} \subseteq \mathcal{A}^{\lambda m + (1 - \lambda) m'}$ for all $m, m' \in \mathbb{R}$ and $\lambda \in {]}0, 1{[}$.*

  (ii) A risk measure ρ is positive homogeneous if and only if the corresponding risk acceptance family $\mathcal{A}$ is positive homogeneous, that is, $\lambda \mathcal{A}^m = \mathcal{A}^{\lambda m}$ for all $m \in \mathbb{R}$ and $\lambda > 0$. Moreover, any positive homogeneous risk order $\preccurlyeq$ satisfies $\lambda \mathcal{L}(x) = \mathcal{L}(\lambda x)$ for all $x \in \mathcal{X}$ and $\lambda > 0$.

  (iii) Any risk measure ρ corresponding to a risk order $\preccurlyeq$ which satisfies $\lambda x \sim x$ for all $\lambda > 0$, is scaling invariant. Moreover, ρ is scaling invariant if and only if the corresponding risk acceptance family $\mathcal{A}$ is scaling invariant, that is, $\lambda \mathcal{A}^m = \mathcal{A}^m$ for all $m \in \mathbb{R}$ and $\lambda > 0$.

  (iv) A risk order $\preccurlyeq$ admits an affine risk measure ρ if and only if it fulfills the Independence and Archimedean properties[22]. This affine risk measure is, up to increasing affine transformations, unique in the class of affine risk measures.

Statement *(iv)* is a well-known result by von Neumann and Morgenstern [53], for the others we refer to Appendix C.2.

Remark 1.14. Even if convexity, positive homogeneity or affinity are not global properties, the scaling invariance though is global since any increasing transformation of a scaling invariant risk measure is scaling invariant. ♦

Example 1.15. The *Sharpe Ratio* introduced by Sharpe [52] is given by

$$\rho(X) := \begin{cases} -\dfrac{E[X]}{\sqrt{E\left[X^2 - E[X]^2\right]}} & \text{if } E[X] > 0 \\ 0 & \text{else} \end{cases}, \quad X \in \mathbb{L}^\infty, \tag{1.9}$$

with convention that $s/0 = -\infty$ for $s < 0$, is quasiconvex and scaling invariant. Even if it is not monotone for the relation "greater than P-almost surely", it is still a scaling invariant risk measure with respect to the trivial preorder. For a monotone alternative to the Sharpe Ratio we refer to [12]. A general study of scaling invariant risk measures can be found in [21].

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[22] A risk order $\preccurlyeq$ satisfies the Independence property if $x \prec y$ implies $\lambda x + (1 - \lambda) z \prec \lambda y + (1 - \lambda) z$ for all $z \in \mathcal{X}$ and $\lambda \in {]}0, 1{[}$ and the Archimedian property if $x \prec z \prec y$ implies the existence of $\lambda, \beta \in {]}0, 1{[}$ such that $\lambda x + (1 - \lambda) y \prec z \prec \beta x + (1 - \beta) y$.

In the spirit of expected utilities by Savage [51], we define the *expected loss* of a random variable as

$$\rho(X) := E_Q[l(-X)], \quad X \in \mathbb{L}^\infty, \tag{1.10}$$

for some probability measure Q absolutely continuous with respect to P and a continuous loss function $l : \mathbb{R} \to \mathbb{R}$. It is a convex risk measure if l is nondecreasing and convex. The expected loss is not affine on the level of random variables, unless l is affine. However, since it is law invariant, it can also be considered on the level of probability distribution by the identification $Q_X = \mu \in \mathcal{M}_{1,c}$ through

$$\tilde{\rho}(\mu) := \int l(-x)\,\mu(dx), \quad \mu \in \mathcal{M}_{1,c}, \tag{1.11}$$

which is an affine risk measure and corresponds to a von Neumann and Morgenstern [53] representation.

Note that a robust version of (1.10) over more than one probability measure yields the representation of Gilboa and Schmeidler [34]. ◇

1.2 Monetary Risk Orders

Especially for financial applications, it is meaningful to express risk in monetary units by means of a numéraire $\pi \rhd 0$, which often is a risk free bank account $\pi = 1 + r$ for some interest rate $r > -1$. Throughout this section, we assume that $\mathcal{X}$ is a vector space.

Definition 1.16. A risk measure $\rho : \mathcal{X} \to [-\infty, +\infty]$ is *cash additive* if for any $m \in \mathbb{R}$ holds

$$\rho(x + m\pi) = \rho(x) - m.$$

An axiomatic approach to monetary risk measure has first been given by Artzner, Delbaen, Eber, and Heath [4] in terms of coherent[23] monetary risk measures. Föllmer and Schied [29] and Frittelli and Rosazza Gianin [33] generalized them to convex monetary risk measures, which by means of Proposition 1.18 correspond in our terminology to cash additive risk measures.

The cash additivity expresses that $\rho(x)$ is precisely the minimal amount of money which has to be reserved on the risk free bank account π to pull the risk of the position x below the level[24] 0. Here again, the cash additivity is not a global property and we call a risk order *cash additive* if it can be represented by at least one cash additive risk measure, see also Remark 1.11.

Theorem 1.17. *A risk order $\preccurlyeq$ is cash additive if and only if the following two conditions hold*

(i) *Certainty equivalent: for any $x \in \mathcal{X}$ such that $y \prec x \prec z$ for some $y, z \in \mathcal{X}$ there exists a unique $m \in \mathbb{R}$ which satisfies $x \sim m\pi$,*

(ii) *Translation indifference: $x \preccurlyeq y$ implies $x + m\pi \preccurlyeq y + m\pi$ for all $m \in \mathbb{R}$.*

Proof, Appendix C.3.

Furthermore, cash additive risk measures share the property of convexity and a special shape of their risk acceptance family.

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[23] A coherent risk measure ρ is a positive homogeneous cash additive risk measure. By Proposition 1.18, ρ is convex, hence, the positive homogeneity implies that ρ is subadditive, that is, $\rho(x + y) \leq \rho(x) + \rho(y)$.

[24] Indeed, $\rho(x + \rho(x)\pi) = \rho(x) - \rho(x) = 0$ and by monotonicity $\rho(x + m\pi) \leq 0$ for any $m \geq \rho(x)$.

Proposition 1.18. *A risk measure ρ is cash additive if and only if the related risk acceptance family $\mathcal{A}$ satisfies*

$$\mathcal{A}^0 = \mathcal{A}^m + m\pi, \quad \text{for all } m \in \mathbb{R}. \tag{1.12}$$

Furthermore, any cash additive risk measure is automatically convex.

Proof[25], Appendix C.4.

This special shape of the risk acceptance family has a concrete economic interpretation. In the theory of monetary risk measures, $\mathcal{A}^0$ is understood as the set of acceptable positions from a regulating agency's point of view. This agency enforces financial institutions with assets x in the risk class $\mathcal{A}^m$ to reserve a liquid amount of money m on a risk free bank account π to ensure that $x + m\pi$ is acceptable in the sense that it belongs to $\mathcal{A}^0$.

Example 1.19. We here list some examples of cash additive risk measures on $\mathbb{L}^\infty$.

The celebrated *mean variance* risk measure introduced by Markowitz [45],

$$\rho(X) := -E[X] + \frac{\gamma}{2}Var(X), \quad X \in \mathbb{L}^\infty, \tag{1.13}$$

is monotone with respect to the trivial preorder but not with respect to the preorder[26] "greater than P-almost surely".

Given $q \in]0,1[$, the so-called *average value at risk* is defined as

$$AV@R_q(X) := \sup_{Q \in \mathcal{Q}_q} E_Q[-X], \quad X \in \mathbb{L}^\infty, \tag{1.14}$$

where $\mathcal{Q}_q$ is the set of probability measures Q absolutely continuous with respect to P such that $dQ/dP \leq 1/q$. This risk measure is positive homogeneous.

Another prominent example is the *entropic risk measure* given by

$$\rho(X) := \ln\left(E\left[\exp(-X)\right]\right), \quad X \in \mathbb{L}^\infty. \tag{1.15}$$

Finally, an important class is the *optimized certainty equivalent* introduced and studied by Ben-Tal and Teboulle [6, 7] which is defined as

$$\rho(X) := \inf_{m \in \mathbb{R}} \{E[l(m-X)] - m\}, \quad X \in \mathbb{L}^\infty, \tag{1.16}$$

where $l : \mathbb{R} \to [-\infty, +\infty[$ is a lower semicontinuous convex nondecreasing loss function such that $l(0) = 0$ and $1 \in \partial l(0)$. ◇

Recently, El Karoui and Ravanelli [28] pointed out that in the framework of monetary risk measures, the risk free bank account π could also be subject to discounting uncertainty. In consequence, a higher amount of liquidity should be reserved today on the bank account π to ensure that risky positions remain acceptable. For this purpose, they introduced the notion of cash subadditivity[27] for convex risk measures, which has been extended to quasiconvex risk measures in [15].

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[25]The automatic convexity is a well-known result, see [26, 33, 15] and the references therein. However, an argumentation relying on (1.12) is presented there. Furthermore, Cheridito and Kupper [18] showed that real-valued risk measure ρ satisfying $\rho(m\pi) = -m$ for all $m \in \mathbb{R}$ are convex exactly when they are cash additive.

[26]A monotone version with respect to the preorder "greater than P-almost surely" has been studied in [44].

[27]Cash subadditive risk measures also appear naturally as the generators describing the one-step actualisation of dynamic cash additive risk measures for stochastic processes, [see 19, 1].

Definition 1.20. A risk measure ρ on $\mathcal{X}$ is *cash subadditive* if for any $m > 0$ holds

$$\rho(x + m\pi) \geq \rho(x) - m.$$

Here again, it is possible to characterize cash subadditive risk measures by the properties of their related risk acceptance families.

Proposition 1.21. *A risk measure ρ is cash subadditive if and only if the related risk acceptance family $\mathcal{A}$ satisfies*

$$\mathcal{A}^n \subseteq \mathcal{A}^{n+m} + m\pi, \quad \text{for all } m > 0 \text{ and } n \in \mathbb{R}. \tag{1.17}$$

Proof, Appendix C.5

We finally illustrate with one example about *numéraire uncertainty* how monetary risk measures—not necessarily cash additive nor cash subadditive—can be defined by economically motivated risk acceptance families.

Example 1.22. For global acting financial institutions, it is reasonable that regulating agencies require the acceptability of risky positions with respect to a basket of currencies in reason of the different interest rate policies. The financial institutions face here some numéraire uncertainty to assess the risk[28]. Modelling this problem is particularly easy from the risk acceptance family point of view. Indeed, let $\mathcal{A}^0$ be the acceptance set given by the regulating institution and let $\mathcal{N} \subseteq \mathcal{K}$ be a set of possible numéraires, for instance \$, €, £ and ¥. Define

$$\mathcal{A}^m := \left\{ X : X + m\pi \in \mathcal{A}^0 \text{ for all } \pi \in \mathcal{N} \right\} = \bigcap_{\pi \in \mathcal{N}} \left\{ \mathcal{A}^0 - m\pi \right\}, \quad m \in \mathbb{R}.$$

Here, a position is of risk level m if this amount, invested in any of the currencies, pulls the risk of the position within the set of acceptable risky position specified by the regulatory agency. Since $\mathcal{A}$ is obviously a risk acceptance family, it defines a risk measure by means of Theorem 2.6. $\diamond$

2 Robust Representation of Risk Orders

The goal of this section is to provide a dual representation of risk orders, which is the key to get a differentiated interpretation of risk perception depending on the underlying setup. To this end, we however need some topological structure. Hence, we assume that $\mathcal{X}$ is a locally convex topological vector space[29] and denote by $\mathcal{X}^*$ its topological dual space endowed with the weak topology $\sigma(\mathcal{X}^*, \mathcal{X})$. Unless explicitly precised, we denote $x^*, y^*, \ldots$ the elements of the dual space $\mathcal{X}^*$.

We assume that the preorder $\unrhd$ is upper semicontinuous, that is, the cone $\mathcal{K} = \{x \in \mathcal{X} : x \unrhd 0\}$ is $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed. The bipolar theorem[30] states that $x \unrhd y$ exactly when $\langle x^*, x - y \rangle \geq 0$ for all x^* in the polar cone

$$\mathcal{K}^\circ := \{x^* \in \mathcal{X}^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in \mathcal{K}\}, \tag{2.1}$$

which is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed. By $\tilde{\mathcal{K}}$ we denote those elements $\pi \in \mathcal{K}$ which are strictly positive with respect to $\mathcal{K}^\circ$, that is

$$\tilde{\mathcal{K}} = \{\pi \in \mathcal{K} : \langle x^*, \pi \rangle > 0 \text{ for all } x^* \in \mathcal{K}^\circ \setminus \{0\}\}. \tag{2.2}$$

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[28] Note that the question of the interest rate uncertainty is similar, since the regulator requires acceptability then with respect to a set of possible interest rates.

[29] The study of $\mathcal{X}$ as a convex subset of a topological vector space is postponed to Section 2.1.

[30] See for instance [2, Theorem 5.103].

The preorder $\rhd$ is called *regular* if there exists a strictly positive element, that is, $\tilde{\mathcal{K}} \neq \emptyset$. In this case, for any $\pi \in \tilde{\mathcal{K}}$ holds $\mathcal{K}^\circ = \mathbb{R}_+ \mathcal{K}^\circ_\pi$, for the normalized the polar set

$$\mathcal{K}^\circ_\pi = \{x^* \in \mathcal{K}^\circ : \langle x^*, \pi \rangle = 1\}. \tag{2.3}$$

Note that the trivial relation $\rhd$ corresponds to the convex cone $\mathcal{K} = \{0\}$ which is not regular as $\tilde{\mathcal{K}} = \emptyset$.

To illustrate the nature of these new elements, we briefly expose to what they concretely correspond in two of the settings introduced in Section 1.

- **Random Variables:** The vector space of P-almost surely bounded random variables $\mathcal{X} = \mathbb{L}^\infty$ admits the cone $\mathcal{K} = \mathbb{L}^\infty_+$ for the preorder "greater than P-almost surely". Depending on the considered topology we alternatively have:

   1. For the $\|\cdot\|_\infty$-norm, the dual space $\mathcal{X}^* = ba\,(P)$ is the set of bounded finitely additive signed measures on $\mathscr{F}$ absolutely continuous with respect to P. The polar cone is then the set of finitely additive measures denoted by $\mathcal{K}^\circ = ba_+\,(P)$. The preorder is regular since $1 \in \tilde{\mathcal{K}}$, for which the normalized polar set $\mathcal{K}^\circ_1 = \mathcal{M}_{1,f}\,(P)$ is the set of all finitely additive probability measures Q absolutely continuous with respect to P.

   2. For the $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-topology, the dual space is $\mathcal{X}^* = \mathbb{L}^1$. In this case, $\mathcal{K}^\circ = \mathbb{L}^1_+$. Here again, the preorder is regular and by means of the Radon-Nikodým theorem, $\mathcal{K}^\circ_1 = \mathcal{M}_1\,(P)$ is the set of all σ-additive probability measures in $\mathcal{M}_{1,f}\,(P)$.

- **Lotteries:** Let ca_c be the vector space of bounded signed measures with compact support spanned by the lotteries $\mathcal{M}_{1,c}$. On ca_c we consider the $\sigma\,(ca_c, C)$-topology, where C is the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. The dual pairing is given by $\langle f, \mu \rangle = \int f\,d\mu$. The first stochastic order corresponds to the cone

$$\mathcal{K} = \left\{\mu \in ca_c : \int f\,d\mu \geq 0 \text{ for all } f \in \mathcal{K}^\circ\right\},$$

  where $\mathcal{K}^\circ$ is the set of those $f \in C$ which are nondecreasing. Notice that this order is not regular since $\tilde{\mathcal{K}} = \emptyset$.

Definition 2.1. A risk order $\preccurlyeq$ is *lower semicontinuous* if $\mathcal{L}\,(x) = \{y \in \mathcal{X} : y \preccurlyeq x\}$ is closed for all $x \in \mathcal{X}$.

The fact that separable lower semicontinuous risk orders admit lower semicontinuous risk measures is a consequence of the so called gap theorem of Debreu [24, 25].

Proposition 2.2. *A risk order $\preccurlyeq$ is separable and lower semicontinuous if and only if there exists a corresponding lower semicontinuous risk measure ρ. Furthermore, the class of lower semicontinuous risk measures of a lower semicontinuous risk order is stable under lower semicontinuous increasing transformation.*

Proof Appendix C.11. An alternative proof for the first assertion can be found in [9].

Note that any lower semicontinuous separable risk order can be represented by a risk measure which is not lower semicontinuous. Even though, the second assertion states that the class of lower semicontinuous risk measures is stable under lower semicontinuous increasing transformation. It can therefore be seen as a global characteristic in a topological sense.

Remark 2.3. The risk acceptance family $\mathcal{A}$ of a lower semicontinuous risk measure ρ is *closed*, that is, $\mathcal{A}^m$ is closed for all $m \in \mathbb{R}$. Conversely, the risk measure ρ corresponding to a closed risk acceptance family $\mathcal{A}$ is lower semicontinuous. ♦

Aside the numerous technicalities, the core idea of the proof leading to the robust representation of the subsequent Theorem 2.6 is insightful since the risk acceptance family plays a central role. To get an intuition of the objects in play and how they get involved, we informally sketch the key steps of the proof in the special case of random variables. To begin with, by way of relation (1.7), we express the risk measure ρ in terms of its risk acceptance family

$$\rho(X) = \inf\{m \in \mathbb{R} : \rho(X) \le m\} = \inf\{m \in \mathbb{R} : X \in \mathcal{A}^m\}.$$

We now exploit the fact that each of these risk acceptance sets $\mathcal{A}^m$ has the polar representation

$$X \in \mathcal{A}^m \quad \text{if and only if} \quad E_Q[-X] \le \alpha_{\min}(Q, m) \text{ for all probability measures } Q,$$

where $\alpha_{\min}(Q, m) = \sup_{X \in \mathcal{A}^m} E_Q[-X]$ is the so called minimal penalty function[31]. Hence

$$\rho(X) = \inf\{m \in \mathbb{R} : X \in \mathcal{A}^m\} = \inf\{m \in \mathbb{R} : E_Q[-X] \le \alpha_{\min}(Q, m) \text{ for all } Q\}.$$

Without duality gap in interchanging the supremum over Q with the infimum over m, we finally get the robust representation

$$\rho(X) = \sup_Q \inf_m \{m \in \mathbb{R} : E_Q[-X] \le \alpha_{\min}(Q, m)\}$$
$$= \sup_Q R(Q, E_Q[-X]),$$

where R is the left inverse of the nondecreasing function $m \mapsto \alpha_{\min}(Q, m)$, that is

$$R(Q, s) = \inf\{m \in \mathbb{R} : s \le \alpha_{\min}(Q, m)\}.$$

Following this sketch of the proof, we define the *minimal penalty function* of a risk acceptance family $\mathcal{A}$ by

$$\alpha_{\min}(x^*, m) := \sup_{x \in \mathcal{A}^m} \langle x^*, -x \rangle, \quad x^* \in \mathcal{K}^\circ \text{ and } m \in \mathbb{R}. \tag{2.4}$$

Notice that even if the risk acceptance family is right-continuous, the penalty function $m \mapsto \alpha_{\min}(x^*, m)$ is generally neither right nor left-continuous[32].

Definition 2.4. A *risk function* is a mapping $R : \mathcal{K}^\circ \times \mathbb{R} \to [-\infty, +\infty]$, which is nondecreasing and left-continuous in the second argument. The set of risk functions is denoted by $\mathcal{R}$.

The left inverse of the minimal penalty function will be the corner stone of the robust representation and is a specific risk function which belongs to the following class.

---

[31] This terminology was introduced in the theory of monetary risk measures, see [29].

[32] Indeed, consider $\Omega = [0, 1]$ with the Borel σ-algebra $\mathscr{F} = \mathscr{B}_{[0,1]}$ and the Lebesgue measure $P = dx$ and define $\mathcal{A}^m = \emptyset$ for $m < 0$, $\mathcal{A}^m = \{X \in \mathbb{L}^\infty : X1_{[m,1]} \ge 0\}$ for $0 \le m \le 1$ and $\mathcal{A}^m = \mathbb{L}^\infty$ for $m > 1$. Obviously, $\mathcal{A}$ is a closed risk acceptance family, and for $x^* = P$ holds $\alpha_{\min}(x^*, m) = -\infty$ for $m < 0$, $\alpha_{\min}(x^*, m) = 0$ for $m = 0$ and $\alpha_{\min}(x^*, m) = +\infty$ for $m > 0$, which is neither right nor left-continuous.

Definition 2.5. By $\mathcal{R}^{\max}$ we denote the set of those risk functions $R \in \mathcal{R}$ for which

(*i*) R is jointly quasiconcave,

(*ii*) $R\left(\lambda x^*, s\right) = R\left(x^*, s/\lambda\right)$ for all $x^* \in \mathcal{K}^\circ$, $s \in \mathbb{R}$ and $\lambda > 0$,

(*iii*) R has a uniform asymptotic minimum, that is, $\lim_{s \to -\infty} R\left(x^*, s\right) = \lim_{s \to -\infty} R\left(y^*, s\right)$ for all $x^*, y^* \in \mathcal{K}^\circ$,

(*iv*) its right-continuous version, $R^+\left(x^*, s\right) := \inf_{s' > s} R\left(x^*, s'\right)$, is upper semicontinuous in the first argument.

Risk functions in $\mathcal{R}^{\max}$ are referred to as *maximal risk functions*. In most examples, see Section 3, the function $m \mapsto \alpha\left(x^*, m\right)$ is continuous and increasing, in which case the maximal risk function is in fact the true inverse of the minimal penalty function, that is, $R\left(x^*, \cdot\right) = \alpha_{\min}^{-1}\left(x^*, \cdot\right)$.

After this preliminary definitions and notations, we present our robust representation results.

Theorem 2.6 (Robust Representation of Risk Orders). *Any lower semicontinuous risk measure ρ : $\mathcal{X} \to [-\infty, +\infty]$ corresponding to a lower semicontinuous risk order $\preccurlyeq$ has the robust representation*

$$\rho\left(x\right) = \sup_{x^* \in \mathcal{K}^\circ} R\left(x^*, \langle x^*, -x \rangle\right), \quad x \in \mathcal{X}, \tag{2.5}$$

for a unique $R \in \mathcal{R}^{\max}$, which is the left inverse of the minimal penalty function $\alpha_{\min}$.

Conversely, for any $R \in \mathcal{R}$, the function ρ defined by (2.5) is a lower semicontinuous risk measure.

Proof, Appendix C.6.

The terminology "robust" in robust representation has been introduced in the theory of monetary risk measures. In the context of risk orders, the "loss" $-x$ is tested under the evaluation x^* by means of the operation $\langle x^*, -x \rangle$. Since a risk perceiver is not sure which exact evaluation is adequate to compute the expected loss, s/he takes a precautionary estimation by considering all possible evaluations weighted according to their plausibility by a risk function $R\left(x^*, \cdot\right)$. This precautionary estimation, core characteristic of risk perception, justifies henceforth the term "robust" for the representation (2.5).

In the case where the preorder $\rhd$ is regular, we obtain a finer robust representation. Here, in line with Definitions 2.4 and 2.5, the set of normalized risk functions $\mathcal{R}_\pi$ with respect to $\pi \in \tilde{\mathcal{K}}$ consists of those mappings $R : \mathcal{K}_\pi^\circ \times \mathbb{R} \to [-\infty, +\infty]$, which are nondecreasing and left-continuous in the second argument. Moreover, $\mathcal{R}_\pi^{\max}$ is the set of those $R \in \mathcal{R}_\pi$, for which R is jointly quasiconcave, R has a uniform asymptotic minimum on $\mathcal{K}_\pi^\circ$ and R^+ is upper semicontinuous in the first argument.

Theorem 2.7 (Robust Representation in the Regular Case). *Let $\rhd$ be a regular preorder. Any lower semicontinuous risk measure $\rho : \mathcal{X} \to [-\infty, +\infty]$ corresponding to a lower semicontinuous risk order $\preccurlyeq$ has the robust representation*

$$\rho\left(x\right) = \sup_{x^* \in \mathcal{K}_\pi^\circ} R\left(x^*, \langle x^*, -x \rangle\right), \quad x \in \mathcal{X}, \tag{2.6}$$

for a unique $R \in \mathcal{R}_\pi^{\max}$, which is the left inverse of the minimal penalty function $\alpha_{\min}$.

Conversely, for any $R \in \mathcal{R}_\pi$, the function ρ defined by (2.6) is a lower semicontinuous risk measure.

Proof, Appendix C.7.

The one-to-one relation between risk measures ρ and their risk functions $R \in \mathcal{R}^{\max}$ is crucial for the dual classification of risk orders and makes comparative statics meaningful. To this aim, Cerreia-Vioglio,

Maccheroni, Marinacci, and Montrucchio [14] introduced the notion of a *complete duality*, in the sense that there exists a one-to-one relation between functions and their respective dual functions within a specified primal and dual class[33]. They give complete duality results for the class of monotone evenly[34] quasiconvex functions and for different subclasses of it including the upper semicontinuous monotone quasiconvex functions. In the spirit of those results, Theorem 2.7 states the complete duality result for the class of lower semicontinuous quasiconvex functions, which is not treated in [15].

Remark 2.8. The technical assumption of lower semicontinuity is often a direct consequence of the monotonicity, see [47, 8, 50, 20, 27, 17]. Furthermore, by means of Proposition 2.2, lower semicontinuous risk orders can be represented by lower semicontinuous risk measures; a similar statement for evenly quasiconvex risk orders is to our knowledge still open. Finally, for some topologies [11] the lower semicontinuity of the risk order automatically ensures the separability, see Section 3.1 and Section 3.2. ♦

In contrast to [14], Theorem 2.6 holds for functions which are monotone with respect to any preorder including the trivial one which corresponds to $\mathcal{K} = \{0\}$ and which are defined on locally convex topological vector spaces. The representation part of our proofs are in line with Penot and Volle [49, Proposition 3.6 and Theorem 3.8]. However, in [49], the robust representation is stated in terms of elements in $\mathcal{X}^*$ rather than $\mathcal{K}^\circ$ or $\mathcal{K}_\pi^\circ$, and more important, uniqueness considerations and characterizations of the maximal risk function are not treated. For further references on quasiconvex duality theory, we refer to de Finetti [23], Greenberg and Pierskalla [35], Crouzeix [22] and the references therein.

The denomination "maximal" risk function is justified by the following result.

Proposition 2.9. *Suppose that a lower semicontinuous risk measure $\rho : \mathcal{X} \to [-\infty, +\infty]$ admits the robust representations*

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} R\left(x^*, \langle x^*, -x \rangle\right) = \sup_{x^* \in \mathcal{K}^\circ} \tilde{R}\left(x^*, \langle x^*, -x \rangle\right), \quad x \in \mathcal{X}, \tag{2.7}$$

for some risk functions $R, \tilde{R}$ with $R \in \mathcal{R}^{\max}$. Then, R is pointwise greater than $\tilde{R}$, that is

$$R(x^*, s) \geq \tilde{R}(x^*, s), \quad \text{for all } x^* \in \mathcal{X}^* \text{ and } s \in \mathbb{R}. \tag{2.8}$$

Proof, Appendix C.8.

Remark 2.10. Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [14, 15] state a complete duality result between upper semicontinuous risk measures and risk functions which are jointly upper semicontinuous. The lower semicontinuous case is different and is stated in terms of an upper semicontinuity condition for the right-continuous version R^+ of the risk function R. Actually, the property (*iv*) in Definition 2.5 cannot be expressed in terms of a semicontinuity condition for the risk function R as illustrated in Appendix C.9. Furthermore, in Appendix C.9 it is shown that the regularity assumption on the preorder $\unrhd$ in Theorem 2.7 cannot be dropped. ♦

In the following proposition we sum up the impact on the robust representation of additional properties of the risk measure as discussed in Section 1. Similar results have been established in [14] in the context of M-spaces.

---

[33]For instance, the Fenchel-Moreau theorem states a complete duality between proper lower semicontinuous convex functions f and their proper lower semicontinuous convex conjugates f^*.
[34]The level sets are evenly convex, that is, they are the intersection of a family of open half-spaces.

Proposition 2.11. *A lower semicontinuous risk measure ρ with corresponding $R \in \mathcal{R}^{\max}$ is convex, positive homogeneous or scaling invariant if and only if $s \mapsto R(x^*, s)$ is respectively convex, positive homogeneous or scaling invariant for all $x^* \in \mathcal{K}^\circ$.*

If the preorder $\rhd$ is π-regular, a lower semicontinuous risk measure ρ with corresponding $R \in \mathcal{R}_\pi^{\max}$ is cash additive if and only if $R(x^*, s + m) = R(x^*, s) + m$ for all $x^* \in \mathcal{K}_\pi^\circ$ and $s, m \in \mathbb{R}$, in which case $R(x^*, s) = s - \alpha_{\min}(x^*, 0)$. It is cash subadditive if and only if $R(x^*, s - m) \geq R(x^*, s) - m$ for all $x^* \in \mathcal{K}_\pi^\circ$, $s \in \mathbb{R}$ and $m > 0$.

The proof, in Appendix C.10, relies on the properties of $m \mapsto \alpha_{\min}(x^*, m)$ inherited from the risk acceptance family.

2.1 Robust Representation of Risk Orders on Convex Sets

In the Theorems 2.6 and 2.7, we assumed that the risk order is defined on a vector space. This is however not the case for many important settings[35]. To address this problem, we continuously extend the risk order to a vector space and then apply Theorem 2.6. Throughout this subsection we assume that $\mathcal{X}$ is a convex subset of a locally convex topological vector space $\mathcal{V}$ and that the preorder $\rhd$ corresponds to the closed convex cone $\mathcal{K} \subseteq \mathcal{V}$. The main difficulty here beyond the continuous extension is to specify for which set of maximal risk functions the uniqueness result in the robust representation is guaranteed. Indeed, the following example illustrates the fact that $\mathcal{R}^{\max}$ is in general too big to state a uniqueness results for risk orders on convex sets as in Theorem 2.6.

Example 2.12. Consider the risk measure $\rho(x) = 0$ on the convex set $\mathcal{X} = \mathbb{R}^+$ in $\mathcal{V} = \mathbb{R}$ with $\mathcal{K} = \mathcal{K}^\circ = \mathbb{R}^+$. For any $c \in [0, +\infty]$, the risk measure $\rho_c(x) = 0$ for $x \geq 0$ and $\rho(x) = c$ otherwise is an extension of ρ on the real line. Direct inspection[36] yields for all $x^* \in \mathbb{R}^+$ and $s \in \mathbb{R}$,

$$
R_c(x^*, s) := \begin{cases} 0 & \text{if } s \leq 0 \\ c & \text{if } s > 0 \text{ and } x^* > 0 \\ +\infty & \text{if } s > 0 \text{ and } x^* = 0 \end{cases} .
$$

Hence $R_c \in \mathcal{R}^{\max}$ for all $c \in [0, +\infty]$. On the other hand, since $\rho = \rho_c$ on $\mathbb{R}^+$ it follows

$$
\rho(x) = \sup_{x^* \geq 0} R_c(x^*, -x^* x), \quad x \in \mathbb{R}^+,
$$

for any $c \in [0, +\infty]$, showing that the uniqueness statement in $\mathcal{R}^{\max}$ fails. $\diamond$

Definition 2.13. A lower semicontinuous[37] risk order $\preccurlyeq$ on $\mathcal{X}$ is *continuously extensible* if

$$
\overline{\mathcal{L}(x) + \mathcal{K}} \cap \mathcal{X} = \mathcal{L}(x) \quad \text{for all } x \in \mathcal{X}. \tag{2.9}
$$

A risk order $\rho : \mathcal{X} \to [-\infty, +\infty]$ is *continuously extensible* if its risk acceptance family $\mathcal{A}$ fulfills

$$
\overline{\mathcal{A}^m + \mathcal{K}} \cap \mathcal{X} = \mathcal{A}^m \quad \text{for all } m \in \mathbb{R}. \tag{2.10}
$$

Remark 2.14. Actually, this rather technical assumption automatically holds for many convex sets. For instance, if $\mathcal{X}$ is open or compact, or if $\mathcal{K}$ is a subset of $\mathcal{X}$ as in the case of consumptions patterns, then

---

[35]Lotteries or consumption patterns for instance.
[36]Computing first the minimal penalty function and then taking the left inverse.
[37]The considered topology on $\mathcal{X}$ is the relative topology induced by the topology on $\mathcal{V}$.

any lower semicontinuous risk order and any lower semicontinuous risk measure on $\mathcal{X}$ is continuously extensible[38]. ♦

Yet, a lower semicontinuous risk measure corresponding to a continuously extensible lower semicontinuous risk measure is not necessarily continuously extensible. However, an assertion in the spirit of Proposition 2.2 holds.

Proposition 2.15. *A lower semicontinuous risk order $\preccurlyeq$ is separable and continuously extensible if and only if there exists a corresponding continuously extensible lower semicontinuous risk measure ρ. Moreover, the class of continuously extensible risk measures is stable under lower semicontinuous increasing transformation.*

Proof, Appendix C.11.

The set of risk functions for which the uniqueness statement holds involves a stability with respect to a closure operation. The $\mathcal{R}^{\max}$-*closure* of a function $R : \mathcal{K}^{\circ} \times \mathbb{R} \to [-\infty, +\infty]$ denoted by $\mathrm{cl}_{\mathcal{R}^{\max}}(R)$ is the pointwise infimum of those functions in $\mathcal{R}^{\max}$ which dominate R, that is

$$\mathrm{cl}_{\mathcal{R}^{\max}}(R)(x^*, s) = \inf\left\{ \tilde{R}(x^*, s) : \tilde{R} \in \mathcal{R}^{\max} \text{ and } \tilde{R} \geq R \right\}.$$

In fact, this closure is itself an element of $\mathcal{R}^{\max}$ as stated in the following proposition.

Proposition 2.16. *The $\mathcal{R}^{\max}$-closure of a function $R : \mathcal{K}^{\circ} \times \mathbb{R} \to [-\infty, \infty]$ is itself an element of $\mathcal{R}^{\max}$.*

Proof, Appendix C.12.

By use of this closure operation, we now define the set of maximal risk functions on $\mathcal{X}$.

Definition 2.17. By $\mathcal{R}_{\mathcal{X}}^{\max}$ we denote the set of those risk functions $R \in \mathcal{R}^{\max}$ such that[39]

$$R(x^*, s) = \mathrm{cl}_{\mathcal{R}^{\max}}\left(\sup_{y^* \in \mathcal{V}^*} R\left(x^* - y^*, s - \delta_{\mathcal{X}}(y^*) \right) \right) \tag{2.11}$$

where $\delta_{\mathcal{X}}$ denotes the support function of the convex set $-\mathcal{X}$ given by

$$\delta_{\mathcal{X}}(y^*) := \sup_{y \in \mathcal{X}} \langle y^*, -y \rangle, \quad y^* \in \mathcal{V}^*. \tag{2.12}$$

Remark 2.18. Depending on the considered set $\mathcal{X}$ the definition of $\mathcal{R}_{\mathcal{X}}^{\max}$ sometimes simplifies. For instance for lotteries we prove in Section 3.2 that $\mathcal{R}_{\mathcal{M}_{1,c}}^{\max} = \mathcal{R}^{\max}$. ♦

We can now state our main representation result for risk orders on convex sets.

Theorem 2.19. *Let ρ be a lower semicontinuous risk measure corresponding to a continuously extensible lower semicontinuous risk order $\preccurlyeq$ on $\mathcal{X}$. Then, ρ has the robust representation*

$$\rho(x) := \sup_{x^* \in \mathcal{K}^{\circ}} R(x^*, \langle x^*, -x \rangle), \quad x \in \mathcal{X}, \tag{2.13}$$

for a unique $R \in \mathcal{R}_{\mathcal{X}}^{\max}$. Moreover, there exists a unique maximal lower semicontinuous risk measure $\hat{\rho}$ on $\mathcal{V}$, which restricted to $\mathcal{X}$ coincides with ρ.

---

[38]Indeed, for $\mathcal{X}$ open it follows $\overline{\mathcal{L}(x) + \mathcal{K}} \cap \mathcal{X} = \overline{(\mathcal{L}(x) + \mathcal{K}) \cap \mathcal{X}} \cap \mathcal{X} = \overline{\mathcal{L}(x)} \cap \mathcal{X} = \mathcal{L}(x)$. If $\mathcal{X}$ is compact, $\mathcal{L}(x)$ is also compact and thus $\overline{\mathcal{L}(x) + \mathcal{K}} \cap \mathcal{X} = (\mathcal{L}(x) + \mathcal{K}) \cap \mathcal{X} = \mathcal{L}(x)$. Finally, if $\mathcal{K} \subseteq \mathcal{X}$ then $\mathcal{L}(x) + \mathcal{K} \subseteq \mathcal{L}(x)$ and in turn $\overline{\mathcal{L}(x) + \mathcal{K}} \cap \mathcal{X} = \overline{\mathcal{L}(x)} \cap \mathcal{X} = \mathcal{L}(x)$. The same argumentation also holds for a lower semicontinous acceptance family $\mathcal{A}$.

[39]By convention $R(x^*, \cdot) \equiv -\infty$ for all $x^* \notin \mathcal{K}^{\circ}$ and $R(\cdot, -\infty) \equiv -\infty$.

Proof, Appendix C.13.

Remark 2.20. To compute the risk function $R \in \mathcal{R}_{\mathcal{X}}^{\max}$ of a risk measure ρ on $\mathcal{X}$, the strategy is to pick any extension $\tilde{\rho}$ of ρ and compute its maximal risk function $\tilde{R} \in \mathcal{R}^{\max}$. Then, the risk function $R \in \mathcal{R}_{\mathcal{X}}^{\max}$ is given by

$$R\left(x^*, s\right) = \mathrm{cl}_{\mathcal{R}^{\max}}\left(\sup_{y^* \in \mathcal{V}^*} \tilde{R}\left(x^* - y^*, s - \delta_{\mathcal{X}}\left(y^*\right)\right)\right).$$

For instance, coming back to the Example 2.12, the maximal lower semicontinuous risk measure which extends ρ is clearly $\rho_{+\infty}$. Furthermore, $\delta_{\mathbb{R}^+}\left(y^*\right) = 0$ for $y^* \in \mathbb{R}^+$ and $+\infty$ otherwise. Hence, a direct computation yields for any $c \in [0, +\infty]$,

$$\sup_{y^* \in \mathbb{R}} R_c\left(x^* - y^*, s - \delta_{\mathbb{R}^+}\left(y^*\right)\right) = \begin{cases} 0 & \text{if } s \leq 0 \\ +\infty & \text{otherwise} \end{cases} = R_{+\infty}\left(x^*, s\right).$$

Since $R_{+\infty} \in \mathcal{R}^{\max}$, it follows $\mathrm{cl}_{\mathcal{R}^{\max}}\left(R_{+\infty}\right) = R_{+\infty}$ and therefore $R = R_{+\infty}$. ♦

3 Illustrative Settings

In the following subsections, we will illustrate how the robust representation in each particular setting introduced in Section 1 provides the key perspective for a context depending interpretation of risk perception.

3.1 Random Variables

In this subsection, we consider the vector space of random variables $\mathcal{X} = \mathbb{L}^\infty$ with the preorder "$\geq$ than P-almost surely", which corresponds to $\mathcal{K} = \mathbb{L}_+^\infty$. If we endow $\mathbb{L}^\infty$ with the $\|\cdot\|_\infty$-topology, the normalized polar set $\mathcal{K}_1^\circ = \mathcal{M}_{1,f}\left(P\right)$ consists of all finitely additive probability measures. However, probability measures which are not σ-additive are not desirable as they do not have a density and can at most be constructed implicitly by use of the axiom of choice. In order to work with the more tractable set of probability measures $\mathcal{K}_1^\circ = \mathcal{M}_1\left(P\right)$, we endow $\mathbb{L}^\infty$ with the $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-topology. Since this weak topology is coarser than the norm topology, we need an extra condition called *Fatou property*, which ensures a risk order to be $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-lower semicontinous.

Definition 3.1 (Fatou Property). A risk order $\preccurlyeq$ has the Fatou property if and only if for any $X, Y \in \mathbb{L}^\infty$ and any $\|\cdot\|_\infty$-bounded sequence $\left(X_n\right)$ converging P-almost surely to X holds

$$X_n \preccurlyeq Y \text{ for all } n \quad \text{implies} \quad X \preccurlyeq Y. \tag{3.1}$$

By means of [11] and under the assumption we take throughout this subsection that the σ-algebra $\mathscr{F}$ is separable[40], the Fatou property implies that the risk order is automatically separable. The following theorem specializes Theorem 2.7 in the present context.

Theorem 3.2. *Any risk order $\preccurlyeq$ on $\mathbb{L}^\infty$ which has the Fatou property can be represented by a $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-lower semicontinuous risk measure $\rho : \mathbb{L}^\infty \to [-\infty, +\infty]$, with the robust representation*

$$\rho\left(X\right) = \sup_{Q \in \mathcal{M}_1(P)} R\left(Q, E_Q\left[-X\right]\right), \quad X \in \mathbb{L}^\infty, \tag{3.2}$$

---

[40] A σ-algebra is separable if it can be generated by a countable collection of sets.

for a unique risk function $R \in \mathcal{R}_1^{\max}$.

Proof, Appendix C.14.

The robust representation (3.2) shows that risk perception in the context of random variables can be interpreted in terms of *model risk*. Indeed, in face of model uncertainty, a prudent approach is adopted where different probability models for the estimation of the expected losses are taken into account, weighted according to their respective plausibility by means of the risk function R.

For the robust representation of the optimized certainty equivalent in Example 1.19 we refer to [7]. This example includes the average value at risk and the entropic risk measure also from Example 1.19. As for the robust representation of the shortfall risk measure in Example 1.10 we refer to [30, Theorem 4.106]. We next address the robust representation of the certainty equivalent of an expected loss, see Example 1.5.

Example 3.3. Given a proper lower semicontinuous convex nondecreasing loss function[41] $l : \mathbb{R} \to]-\infty, +\infty]$, the certainty equivalent of an expected loss is given by

$$\rho(X) = l^{-1}\left(E\left[l\left(-X\right)\right]\right), \quad X \in \mathbb{L}^\infty.$$

In the following, we assume for simplicity that l is differentiable on its domain. By use of (B.4), it follows that for any $Q \in \mathcal{M}_1(P)$ and $m \in \mathbb{R}$

$$\alpha_{\min}(Q, m) = \sup_{\{X \in \mathbb{L}^\infty \mid E[l(-X)] \leq l^+(m)\}} E_Q[-X] = \sup_{X \in \mathbb{L}^\infty} E\left[-\frac{dQ}{dP}X - \frac{1}{\beta}\left(l\left(-X\right) - l^+\left(m\right)\right)\right],$$
(3.3)

for some Lagrange multiplier $\beta := \beta(Q, m) > 0$. The first order condition yields

$$-\frac{dQ}{dP} + \frac{1}{\beta}l'\left(-\hat{X}\right) = 0.$$

The derivative l' is nondecreasing. We denote by h its right inverse. Assume then that $\hat{X} = -h\left(\beta dQ/dP\right)$ fulfills the previous equation[42]. Then, under integrability and positivity conditions, β is determined through the equation

$$E\left[l\left(h\left(\beta\frac{dQ}{dP}\right)\right)\right] = l^+(m).$$
(3.4)

Plugging the optimizer $\hat{X}$ in (3.3) yields

$$\alpha_{\min}(Q, m) = E_Q\left[h\left(\beta\frac{dQ}{dP}\right)\right],$$
(3.5)

the left inverse of which finally delivers R. We subsequently list closed form solutions for specific loss functions.

- **Quadratic Function:** Let $l(s) = s^2/2 + s$ for $s \geq -1$ and $-1/2$ otherwise for which $E\left[l\left(-X\right)\right]$ corresponds to a monotone version of the mean-variance risk measure Markowitz [45]. For $m \geq -1$, since $1 \in \mathcal{A}^m$ holds $\alpha_{\min}(Q, m) = -E_Q[1] = -1$. Otherwise, the first order condition

---

[41]In Example 1.5, l was increasing. Here, since l is nondecreasing, its left inverse l^{-1} is nondecreasing and lower semicontinuous, see Appendix B.

[42]This is often the case, for instance if l' is increasing.

yields $\hat{X} = \beta dQ/dP - 1$ and therefore

$$\alpha_{\min}(Q, m) = (1 + m) E\left[(dQ/dP)^2\right]^{1/2} - 1.$$

Hence,

$$R(Q, s) = \frac{s + 1}{\|dQ/dP\|_{\mathbb{L}^2}} - 1, \quad Q \in \mathcal{M}_1(P),$$

if $s > -1$ and $-\infty$ otherwise.

- **Exponential Function:** For $l(s) = e^s - 1$, corresponding to the entropic risk measure, it follows

$$R(Q, s) = s - E\left[\frac{dQ}{dP} \log\left(\frac{dQ}{dP}\right)\right], \quad Q \in \mathcal{M}_1(P).$$

- **Logarithm Function:** If $l(s) = -\ln(-s)$ for $s < 0$ and $l = +\infty$ elsewhere, then

$$R(Q, s) = s \exp\left(-E\left[\ln\left(\frac{dQ}{dP}\right)\right]\right), \quad Q \in \mathcal{M}_1(P).$$

- **Power Function:** If $l(s) = -(-s)^{1-\gamma}/(1 - \gamma)$ for $s \le 0$ and $l = +\infty$ elsewhere for $0 < \gamma < 1$, we obtain

$$R(Q, s) = sE\left[(dQ/dP)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{1-\gamma}}, \quad Q \in \mathcal{M}_1(P). \qquad \diamond$$

As for the economic index of riskiness, we use the same technique.

Example 3.4. For the definition and notations, we refer to Example 1.10. The risk acceptance family is given for $m > 0$ by $\mathcal{A}^m = \{X \in \mathbb{L}^\infty : E[l(-X/m)] \le c_0\}$, and for $m \le 0$ by $\mathcal{A}^m = \mathbb{L}_+^\infty$. Applying the same technique as for the certainty equivalent yields

$$\alpha_{\min}(Q, m) = \sup_{X \in \mathbb{L}^\infty} E\left[-\frac{dQ}{dP}X - \frac{1}{\beta}\left(l\left(-\frac{X}{m}\right) - c_0\right)\right] = E_Q\left[mh\left(\beta m\frac{dQ}{dP}\right)\right], \quad Q \in \mathcal{M}_1(P),$$

where h is a pseudo inverse of l' and the Lagrange multiplier is given by $E[l(h(\beta m \, dQ/dP))] = c_0$. In the case of Aumann and Serrano [5] where $l(s) = e^s$ and $c_0 > 1$, holds

$$R(Q, s) = \frac{s}{E_Q[\ln(c_0 dQ/dP)]}, \quad Q \in \mathcal{M}_1(P).$$

In the case of Foster and Hart [32] where $l(s) = -\ln(1 - s)$ and $c_0 > 0$ holds

$$R(Q, s) = \frac{s}{1 - \exp\left(E\left[\ln\left(\frac{dQ}{dP}\right)\right] - c_0\right)}, \quad Q \in \mathcal{M}_1(P)$$

and due to Proposition 2.11, both are positive homogeneous. $\qquad \diamond$

3.2 Lotteries

As for the risk perception in the setting of lotteries with compact support $\mathcal{X} = \mathcal{M}_{1,c}$ we consider the induced $\sigma(ca_c, C)$-topology, where ca_c denotes the vector space of signed measures with compact support.

Recall that the first stochastic order is determined by the polar cone

$$\mathcal{K}^{\circ} = \{f \in C : f \text{ is nondecreasing}\}.$$

Even though $\mathcal{M}_{1,c}$ is not a compact subset of ca_c, we obtain a robust representation from Theorem 2.19 with the remarkable facts that the separability of the risk order is a consequence of the lower semicontinuity and that $\mathcal{R}^{\max}_{\mathcal{M}_{1,c}} = \mathcal{R}^{\max}$.

Theorem 3.5. *Any lower semicontinuous risk order $\preccurlyeq$ on $\mathcal{M}_{1,c}$ which is monotone with respect to the first stochastic order can be represented by a lower semicontinuous risk measure $\rho : \mathcal{M}_{1,c} \to [-\infty, +\infty]$, which has the robust representation*

$$\rho(\mu) = \sup_{l \in \mathcal{K}^{\circ}} R\left(l, \int l(-x)\, \mu(dx)\right), \quad \mu \in \mathcal{M}_{1,c}, \tag{3.6}$$

for a unique $R \in \mathcal{R}^{\max} = \mathcal{R}^{\max}_{\mathcal{M}_{1,c}}$.

Proof, Appendix C.15.

A related representation on the level of simple lotteries has been studied independently by Cerreia-Vioglio [13]. As a consequence of the monotonicity, we provide in Delbaen, Drapeau, and Kupper [27] and Cheridito, Drapeau, and Kupper [17] automatic continuity results for affine and general risk orders respectively, allowing to drop or weaken the lower semicontinuity assumption in Theorem 3.5.

In (3.6) the loss of a lottery μ is tested with respect to a nondecreasing continuous loss function l. However, the risk induced by the uncertainty about the reliability of this loss function yields a precautious approach by means of the risk function R. Hence, risk perception on the level of lotteries might be viewed as *distributional risk*.

Note that the certainty equivalent of the expected loss of a lottery introduced in Example 1.5 is already in its robust representation form[43].

Example 3.6. Following the prescriptions of Basel II, "Value at Risk" is the central instrument used by banking institutions to assess their exposure to risk in a monetary way. Regardless of repeated critics, starting with Artzner, Delbaen, Eber, and Heath [4], that it might penalize diversification since it is not quasiconvex, this measure instrument remains astonishingly resilient in the practice. There are several arguments for the defense of the "Value at Risk"[44], but the most recurrent one is that many persons think that it gives some indications about risk. It is this strong but erroneous belief we here want to study and try to explain.

The "Value at Risk" is defined for $q \in]0, 1[$ by

$$V@R_q(X) = \sup\{s \in \mathbb{R} : P[X + s \le 0] > q\}, \quad X \in \mathbb{L}^{\infty}. \tag{3.7}$$

This functional is cash additive and monotone, but not quasiconvex. From its definition, $V@R_q$ depends only on the distribution of X, and can therefore be viewed[45] on $\mathcal{M}_{1,c}$ for $I = \mathbb{R}$, that is

$$V@R_q(\mu) := \sup\{s \in \mathbb{R} : \mu(]-\infty, -s]) > q\} = -F_{\mu}^{-1}(q), \quad \mu \in \mathcal{M}_{1,c}. \tag{3.8}$$

where F_{μ}^{-1} is the right inverse of the nondecreasing function $s \mapsto F_{\mu}(s) := \mu(]-\infty, s])$. In fact,

---

[43] Indeed for $\rho(\mu) = l_0^{-1}\left(\int l_0(-x)\, \mu(dx)\right)$, is $R(l, t) = l_0^{-1}(t)$, if $l = l_0$, and $\inf_{\mu \in \mathcal{M}_{1,c}} \rho(\mu)$ otherwise.

[44] For instance, restricted to Gaussian risky assets, the "Value at Risk" is a convex risk measure.

[45] This has been done in [54] for the case of monetary risk measures considered on the level of probability distributions.

$V@R_q(X) = V@R_q(\mu)$ for $\mu = P_X \in \mathcal{M}_{1,c}$. On the level of probability distributions, $V@R_q$ is monotone with respect to the first stochastic order. Moreover, for any risk level $m \in \mathbb{R}$, it follows from relation (B.4) that

$$
\begin{aligned}
\mathcal{A}^m &= \left\{ \mu \in \mathcal{M}_{1,c} : V@R_q(\mu) \le m \right\} = \left\{ \mu \in \mathcal{M}_{1,c} : F_\mu^{-1}(q) \ge -m \right\} \\
&= \left\{ \mu \in \mathcal{M}_{1,c} : q \ge F_\mu^-(-m) \right\} = \left\{ \mu \in \mathcal{M}_{1,c} : q \ge \mu\left(]-\infty, -m[\right) \right\}, \quad (3.9)
\end{aligned}
$$

which is a convex set. Therefore, $V@R_q$ is a risk measure on $\mathcal{M}_{1,c}$.

In Appendix C.16, we further show that $V@R_q$ is lower semicontinuous and provide the following robust representation

$$
V@R_q(\mu) = \sup_{l \in \mathcal{K}^\circ} l^{-1} \left(\frac{\int l(-x)\,\mu(dx) - q l(+\infty)}{1-q} \right). \tag{3.10}
$$
$$\diamond$$

From this viewpoint, the strong belief of the finance industry in the "Value at Risk" as a risk measure, is truly founded since it is indeed a risk measure on the level of probability distributions. Yet, it is a fundamental error to consider it as a reliable instrument to assess the risk of financial positions which are definitively random variables for which a scenario-wise diversification is needed. So, even if this instrument is in principle a sound one, it is fundamentally misused in the wrong environment.

3.3 Consumption Patterns

Consumption patterns of a commodity reveal another form of uncertainty, namely the one driven by the perception of different future points in time at which this consumption occurs. The risk perception induced by this intertemporal dimension will be the subject of this subsection.

Recall that the commodity space $\mathcal{X} = \mathcal{CS}$ is the set of nondecreasing right-continuous functions $c : [0,1] \to \mathbb{R}$. In a stimulating discussion, Hindy, Huang, and Kreps [37] gave several reasons why the Orlicz topology induced by the Luxemburg norm $\|\cdot\|_\eta$ on the Orlicz heart[46] $\mathcal{V} = \mathcal{V}_\eta \supset \mathcal{CS}$ is economically and mathematically reasonable to address coherently both continuity and jumps issues for preferences over consumption patterns. As for the preorder we consider $\mathcal{K}$ as the set of nondecreasing elements[47] in $\mathcal{V}$ with polar cone

$$
\mathcal{K}^\circ = \left\{ f \in \mathcal{V}_\eta^* : \int_0^1 f_s c_s\, ds + f_1 c_1 \ge 0 \text{ for all } c \in \mathcal{K} \right\}.
$$

By use of integration by parts it follows that for any $c \in \mathcal{CS}$ and $f \in \mathcal{V}_\eta^*$ the linear pairing is given by

$$
\langle f, c \rangle = \int_0^1 f_s c_s\, ds + f_1 c_1 = \int_{0-}^1 \beta_s\, dc_s, \tag{3.11}
$$

where $\beta_t = \beta_t(f) = \int_t^1 f_s\, ds + f_1$. We can therefore identify $f \in \mathcal{V}_\eta^*$ with the respective $\beta = \beta(f)$,

---

[46]The space $\mathcal{CS}$ is a subspace of the Orlicz heart $\mathcal{V}_\eta$ consisting of all measurable functions $c : [0,1] \to \mathbb{R}$ such that $\int_0^1 \eta(m\,|c_t|)\, dt + \eta(m\,|c_1|) < +\infty$ for all $m > 0$, where $\eta : [0,+\infty[\to [0,+\infty[$ is a convex function with $\eta(0) = 0$ and $\lim_{t \to +\infty} \eta(t)/t = +\infty$. Equipped with the topology induced by the Luxemburg norm $\|c\|_\eta = \inf\{m > 0 : \int_0^1 \eta(|c_t|/m)\, dt + \eta(|c_1|/m) \le 1\}$, the dual space $\mathcal{V}_\eta^*$ is the Orlicz space consisting of all measurable functions $f : [0,1] \to \mathbb{R}$ with finite Luxemburg norm $\|\cdot\|_\nu$, where ν is the convex conjugate of η.

[47]In which case $\mathcal{K} \cap \mathcal{CS} = \mathcal{CS}$, and thus corresponds to the preorder introduced in Section 1.

for which the linear pairing modifies to $\int_{0-}^{1} \beta_s dc_s$. Since $\int_{0-}^{1} \beta_s dc_s \geq 0$ for all $c \in \mathcal{CS}$ is equivalent to $\beta_s \geq 0$, the polar cone $\mathcal{K}^\circ$ can then be identified with

$$\mathcal{D} = \left\{ \beta = \beta(f) : f \in \mathcal{V}_\eta^*, \; \beta \geq 0 \right\}. \tag{3.12}$$

Theorem 3.7. *Any lower semicontinuous risk measure ρ of a lower semicontinuous risk order $\preccurlyeq$ on $\mathcal{CS}$ monotone with respect to $\mathcal{K} = \mathcal{CS}$ has the robust representation*

$$\rho(c) = \sup_{\beta \in \mathcal{D}} R\left(\beta, -\int_{0-}^{1} \beta_t dc_t \right), \quad c \in \mathcal{CS}, \tag{3.13}$$

for a unique maximal risk function $R \in \mathcal{R}_{\mathcal{CS}}^{\max}$.

Proof, Appendix C.17.

Diversification on the level of consumption patterns typically avoids concentration effects of consumption at particular times. The interpretation excerpted by the robust representation is that risk orders in this context address a discounting estimation. The value $\int_{0-}^{1} \beta_s dc_s$ represents the discounted value of the consumption pattern c for the discounting factor β which gives different weights to different points in time. The higher this discounted value the less risky the consumption pattern. The uncertainty arising from the choice of an adequate discounting factor is then addressed in a precautionary way by means of the risk function R and justifies here an interpretation of risk perception as a *discounting risk*. We illustrate this intertemporal perspective in risk perception with a class of risk measure inspired by [37]

$$\rho(c) = \int_{0}^{1} l\left(t, -\int_{t-k_1(t)}^{t+k_2(t)} \theta(t,s) dc_s \right) dt, \quad c \in \mathcal{CS}, \tag{3.14}$$

for some parameter function θ satisfying $\theta(t,s) = 0$ whenever $s \notin [0,1]$ and which is jointly continuous on $\mathbb{R} \times [0,1]$. The functions k_1 and k_2 are continuous and $l : \mathbb{R}^2 \to \mathbb{R}$ is a jointly measurable function which is continuous in the first argument and lower semicontinuous nondecreasing and convex in the second argument. The terminology intertemporal risk measure means that risk estimation of a commodity pattern at time t is not the "instant" consumption dc_t but a weighted average in the time around t.

Proposition 3.8. *The function ρ given by (3.14) is a lower semicontinuous risk measure which is monotone with respect to $\mathcal{K} = \mathcal{CS}$.*

For a Proof, see Appendix C.18.

Example 3.9. In the following we address the robust representation for the risk measure

$$\rho(c) = \int_{0}^{1} l\left(-\int_{0-}^{t} e^{-\gamma(t-s)} dc_s \right) dt, \tag{3.15}$$

that is, $\theta(t,s) = e^{-\gamma(t-s)}$ and $l : \mathbb{R} \to]-\infty, +\infty]$ is a lower semicontinuous convex function which is increasing and continuously differentiable on $]-\infty, 0[$. After computations involving a relaxation technique to the whole vector space $\mathcal{V}_\eta$ given in the Appendix C.19, we obtain the following risk functions

- $l(x) = e^x$ yields $\tilde{R}(\beta, s) = \exp\left(\dfrac{s + \ln\left(\int_0^1 \Delta\beta_t dt \right) \int_0^1 \Delta\beta_t dt - \int_0^1 \ln(\Delta\beta_t)\Delta\beta_t dt}{\int_0^1 \Delta\beta_t dt} \right)$ and

- $l(x) = -\ln(-x)$ yields $\tilde{R}(\beta, s) = -\ln(-s) + \int_0^1 \ln(\Delta \beta_t) \, dt$.

where $\beta \in \mathcal{D}$ with $\Delta \beta_t \geq 0$ and $\beta_1 = 0$, and otherwise holds $\tilde{R} = -\infty$. Even if $\tilde{R}$ represents also ρ, the maximal risk function R corresponding to ρ in the sense of Theorem 2.19 is given by

$$R(\beta, s) := cl_{\mathcal{R}^{\max}} \sup_{\bar{\beta} \in \mathcal{D}} \tilde{R}(\beta - \bar{\beta}, s), \tag{3.16}$$

since $\delta_{\mathcal{CS}}(\bar{\beta}) = 0$ if $\bar{\beta} \in \mathcal{D}$ and $+\infty$ otherwise. $\diamond$

3.4 Stochastic Kernels

Due to their intrinsic mixed nature between random variables and lotteries, stochastic kernels illustrate the interplay of risk perception between model risk and distributional risk. They typically corresponds to the setting of Anscombe and Aumann [3] used in [34, 39, 43, 16, 31]. Recall that given a probability space[48] $(\Omega, \mathscr{F}, P)$, the set of stochastic kernels denoted by $\mathcal{SK}$ is the set of measurable functions $\tilde{\mu} : \Omega \to \mathcal{M}_{1,c}$ for which there exists[49] $m > 0$ such that $\tilde{\mu}(\cdot, [-m, m]) = 1$ P-almost surely. As for the monotonicity, $\tilde{\mu} \trianglerighteq \tilde{\nu}$ if $\tilde{\mu}(\omega)$ dominates $\tilde{\nu}(\omega)$ in the second stochastic order for P-almost all $\omega \in \Omega$. The set of lotteries with compact support $\mathcal{M}_{1,c}$ can be identified with the set of P-almost surely constant elements of $\mathcal{SK}$. Further, the P-almost surely bounded random variables are canonically embedded into $\mathcal{SK}$ by means of the relation $X \mapsto \tilde{\mu} := \delta_X$ where δ_X is defined as the stochastic kernel equal to the Dirac measure at the point $X(\omega)$, that is, $\tilde{\mu}(\omega) = \delta_{X(\omega)}$ for almost all $\omega \in \Omega$. We henceforth say that a risk order $\preccurlyeq$ on $\mathcal{SK}$ satisfies the Fatou property if for any $\|\cdot\|_\infty$-bounded sequence X_n of random variables converging P-almost surely to a bounded random variable X and any bounded random variable Y holds

$$\delta_{X_n} \preccurlyeq \delta_Y \text{ for all } n \quad \text{implies} \quad \delta_X \preccurlyeq \delta_Y.$$

The adequate condition which separates a risk order $\preccurlyeq$ on $\mathcal{SK}$ in a model risk and a distributional risk component is given by

$$\tilde{\mu}(\omega) \preccurlyeq \tilde{\nu}(\omega) \text{ for } P\text{-almost all } \omega \in \Omega \quad \text{implies} \quad \tilde{\mu} \preccurlyeq \tilde{\nu}. \tag{3.17}$$

Theorem 3.10. *Let $\preccurlyeq$ be risk order on $\mathcal{SK}$ such that*

(i) $\preccurlyeq$ fulfills the Fatou property,

(ii) $\delta_s \prec \delta_t$ for any two reals s, t with $s > t$,

(iii) $\preccurlyeq$ restricted to $\mathcal{M}_{1,c}$ is $\sigma(\mathcal{M}_{1,c}, C)$-lower semicontinuous and sensitive, that is

$$\delta_c \prec \mu \text{ for some } c \in \mathbb{R} \quad \text{implies} \quad \delta_{c-\varepsilon} \preccurlyeq \mu \text{ for some } \varepsilon > 0, \tag{3.18}$$

(iv) $\preccurlyeq$ satisfies condition (3.17).

Then, $\preccurlyeq$ can be represented by a risk measure ρ which factorizes into a model risk component and a distributional risk component, that is

$$\rho(\tilde{\mu}) = \Phi\left(\omega \mapsto -g\left(\tilde{\mu}(\omega)\right)\right), \quad \tilde{\mu} \in \mathcal{SK}, \tag{3.19}$$

---

[48]In line with Subsection 3.1 we assume that the σ-algebra $\mathscr{F}$ is separable.
[49]The constant $m > 0$ depends on the choice of $\tilde{\mu}$.

where $\Phi : \mathbb{L}^\infty \to \mathbb{R}$ *is a* $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$*-lower semicontinuous risk measure and* $g : \mathcal{M}_{1,c} \to \mathbb{R}$ *is a* $\sigma\left(\mathcal{M}_{1,c}, C\right)$*-lower semicontinuous risk measure such that* $\Phi(c) = g\left(\delta_c\right) = -c$ *for all* $c \in \mathbb{R}$.

Conversely, any risk order corresponding to a risk measure of the form (3.19) *fulfills the conditions (i) to (iv).*

Proof, see Appendix C.20.

In contrast to the representations provided in [34, 39, 43, 16] the risk order in Theorem 3.10 restricted to the set of lotteries $\mathcal{M}_{1,c}$ is not necessarily affine.

Remark 3.11. By means of Theorem 3.2 and Theorem 3.5, any risk measure of the form (3.19) has the robust representation[50]

$$\rho\left(\tilde{\mu}\right) = \sup_{Q \in \mathcal{M}_1(P),\, l \in \mathcal{K}^{1,\circ}} R\left(Q, E_Q\left[r\left(l, \int l\left(-s\right)\tilde{\mu}\left(\cdot, ds\right)\right)\right]\right), \quad \tilde{\mu} \in \mathcal{SK}, \tag{3.20}$$

where R and r are the maximal risk functions of Φ and g, respectively. ◆

A Notations and Basic Concepts

Throughout, the *extended real line* $[-\infty, +\infty] := \mathbb{R} \cup \{-\infty, +\infty\}$ is considered with the canonical order and the convention $+\infty + (-\infty) = +\infty$. The extended real line endowed with the metric $d(x, y) := \arctan(|x - y|)$ is a compact Polish space[51]. A function $f : \mathcal{X} \to [-\infty, +\infty]$, where $\mathcal{X} \subseteq \mathcal{V}$ is a convex subset of a vector space $\mathcal{V}$, is

- *convex* if $f\left(\lambda x + (1 - \lambda) y\right) \leq \lambda f\left(x\right) + (1 - \lambda) f\left(y\right)$ for all $x, y \in \mathcal{X}$ and $\lambda \in \,]0, 1[$, *concave* if $-f$ is convex, and *affine* if f is concave and convex,

- *quasiconvex* if $f\left(\lambda x + (1 - \lambda) y\right) \leq \max\{f\left(x\right), f\left(y\right)\}$ for all $x, y \in \mathcal{X}$ and $\lambda \in \,]0, 1[$, *quasiconcave* if $-f$ is quasiconvex, and *quasiaffine* if f is quasiconvex and quasiconcave.

A convex function $f : \mathcal{X} \to [-\infty, +\infty]$ is *proper* if $f > -\infty$ and $f(x) \in \mathbb{R}$ for some $x \in \mathcal{X}$. A concave function $f : \mathcal{X} \to \mathbb{R}$ is *proper* if $-f$ is proper.

For a nondecreasing function $f : [-\infty, +\infty] \to [-\infty, +\infty]$, we denote by f^- and f^+ the respective unique left- and right-continuous versions of f, that is,

$$f^-\left(s\right) = \sup_{t < s} f\left(t\right) \quad \text{and} \quad f^+\left(s\right) = \inf_{t > s} f\left(t\right), \quad s \in \mathbb{R}, \tag{A.1}$$

which satisfy $f^- \leq f \leq f^+$ with convention that $f^-\left(-\infty\right) = -\infty$ and $f^+\left(+\infty\right) = +\infty$. Note that f^- and f^+ only differ on a countable subset of $\mathbb{R}$.

If $\mathcal{X}$ is a topological space, a function $f : \mathcal{X} \to [-\infty, +\infty]$ is called *lower semicontinuous* if $\{x \in \mathcal{X} : f\left(x\right) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$ and *upper semicontinuous* if $-f$ is lower semicontinuous. The Fenchel-Legendre conjugate f^* of a function f is defined as

$$f^*\left(x^*\right) := \sup_{x \in \mathcal{X}}\{\langle x^*, x\rangle - f\left(x\right)\}, \quad x^* \in \mathcal{X}^*, \tag{A.2}$$

whereby $\mathcal{X}^*$ is the topological dual of $\mathcal{X}$ and with the convention that $\sup \emptyset = \inf \mathbb{R} = -\infty$ and $\sup \mathbb{R} = \inf \emptyset = \infty$.

The *infimal convolution* of convex functions $f_1, f_2 : \mathcal{X} \to [-\infty, +\infty]$ is defined as

$$f_1 \square f_2(x) := \inf_{x_1 + x_2 = x}\{f_1\left(x_1\right) + f_2\left(x_2\right)\}, \quad x \in \mathcal{X}.$$

---

[50] Since the risk measure g is monotone with respect to the second stochastic order, it is also monotone with respect to the first stochastic order.

[51] Recall that a Polish space is a separable complete metrisable space.

By first part of Theorem 16.4 in [Rockafellar], which carries over to the present context, holds

$$\mathrm{cl}\,(f_1 \square f_2) = (f_1^* + f_2^*)^* . \tag{A.3}$$

B Pseudo Inverse

We here present the notion of the pseudo, left and right inverse of a nondecreasing function.

Definition B.1. A function $g : [-\infty, +\infty] \to [-\infty, +\infty]$ is a pseudo inverse of a nondecreasing function $f : [-\infty, \infty] \to [-\infty, +\infty]$ if

$$f^-\left(g\left(t\right)\right) \le t \le f^+\left(g\left(t\right)\right), \quad t \in [-\infty, +\infty] . \tag{B.1}$$

The *left inverse* $f^{(-1,l)}$ and the *right inverse* $f^{(-1,r)}$ are defined as

$$f^{(-1,l)}\left(t\right) := \sup\{s \in \mathbb{R} : f\left(s\right) < t\} = \inf\{s \in \mathbb{R} : f\left(s\right) \ge t\}, \quad t \in [-\infty, +\infty] , \tag{B.2}$$

$$f^{(-1,r)}\left(t\right) := \sup\{s \in \mathbb{R} : f\left(s\right) \le t\} = \inf\{s \in \mathbb{R} : f\left(s\right) > t\}, \quad t \in [-\infty, +\infty] . \tag{B.3}$$

The definition of the pseudo inverse carries over to nondecreasing functions f defined on some interval $I \subseteq [-\infty, +\infty]$, by considering the extension also denoted by f given by $f\left(x\right) = \sup_{y \in I} f\left(y\right)$ for $x > I$ and $f\left(x\right) = \inf_{y \in I} f\left(y\right)$ for $x < I$. The following proposition summarises known results on pseudo inverses, see also Penot and Volle [48], Föllmer and Schied [30].

Proposition B.2. *Given a nondecreasing function* $f : [-\infty, +\infty] \to [-\infty, +\infty]$, *any pseudo inverse* g *of* f *is nondecreasing,* $f^{(-1,l)} = g^- \le g \le g^+ = f^{(-1,r)}$ *and all pseudo inverses of* f *differ at most on a countable subset of* $[-\infty, +\infty]$. *Furthermore,* f *is itself a pseudo inverse of any of its pseudo inverses.*
If f *is moreover left-continuous, then* $g^{(-1,l)} = f$ *for any pseudo inverse* g *of* f *and*

$$f\left(s\right) \le t \quad \text{if and only if} \quad s \le f^{(-1,r)}\left(t\right). \tag{B.4}$$

Symmetrically, if f *is right-continuous, then* $g^{(-1,r)} = f$ *for any pseudo inverse* g *of* f *and*

$$f\left(s\right) \ge t \quad \text{if and only if} \quad s \ge f^{(-1,l)}\left(t\right). \tag{B.5}$$

Finally, if f *is right-continuous, then* f *is concave if and only if* $f^{(-1,l)}$ *is convex.*

Proof. Consider a nondecreasing function $f : [-\infty, +\infty] \to [-\infty, +\infty]$ and a pseudo inverse g of f. By definition, $f^{(-1,l)} \le g^- \le g \le g^+ \le f^{(-1,r)}$. Fix now a decreasing sequence $t_n \searrow t \in [-\infty, +\infty[$. Then

$$\{s \in \mathbb{R} : f\left(s\right) > t\} = \bigcup_{n \in \mathbb{N}} \{s \in \mathbb{R} : f\left(s\right) > t_n\},$$

and therefore $f^{(-1,r)}\left(t_n\right) \searrow f^{(-1,r)}\left(t\right)$ for any $t < +\infty$. Hence, since $f^{(-1,r)}\left(+\infty\right) = +\infty$ by definition, $f^{(-1,r)}$ is right-continuous. The fact that $g^+ = f^{(-1,r)}$ is immediate as they only differ on the countable set of their respective discontinuities and both are right-continuous. A similar argumentation yields $f^{(-1,l)}$ is left-continuous and $f^{(-1,l)} = g^-$.

For any pseudo inverse g of f holds $g\left(t\right) \ge s$ whenever $t > f\left(s\right)$ and therefore $g^+\left(f\left(s\right)\right) \ge s$. Conversely, $g\left(t\right) \le s$ whenever $t < f\left(s\right)$ and thus $g^-\left(f\left(s\right)\right) \le s$, that is, f is a pseudo inverse of g. In particular, if f is left continuous, respectively right continuous, then $g^{(-1,l)} = f$, respectively $g^{(-1,r)} = f$.

Further, the definition of $f^{(-1,l)}$ and $f^{(-1,r)}$ imply the implications "$\Rightarrow$" of relations (B.4) and (B.5). The reverse implications "$\Leftarrow$" follow from $(f^{(-1,r)})^{(-1,l)} = f$ in case that f is left-continuous and $(f^{(-1,l)})^{(-1,r)} = f$ if f is right-continuous.

Finally, if f is right-continuous, from (B.5) follows that the hypograph of f and the epigraph of $f^{(-1,l)}$ are related to each other by means of the relation: $(s,t) \in hypo\left(f\right)$ if and only if $(t,s) \in epi\left(f^{(-1,l)}\right)$. Hence, the last assertion follows. $\qquad\square$

C Technical Proofs

C.1 Proof of Theorem 1.7: It is straightforward to check that $\preccurlyeq$ is a risk order if and only if $\rho_\preccurlyeq$ is a risk measure, that $\preccurlyeq_{\rho_\preccurlyeq} = \preccurlyeq$, and that ρ and $\rho_{\preccurlyeq_\rho}$ coincide up to an increasing transformation. It remains to show the one-to-one relation between risk measures and risk acceptance families.

Step 1. Let ρ be a risk measure with corresponding level sets

$$\mathcal{A}_\rho^m := \{x \in \mathcal{X} : \rho(x) \leq m\}, \quad m \in \mathbb{R}.$$

Then, $\mathcal{A}_\rho^m \subseteq \mathcal{A}_\rho^n$ for any $m \leq n$ which together with the monotonicity of ρ implies the monotonicity of $\mathcal{A}_\rho$. Since level sets of quasiconvex functions are convex it follows that $\mathcal{A}_\rho$ is convex. Obviously, $\mathcal{A}_\rho^m \subseteq \bigcap_{n>m} \mathcal{A}_\rho^n$ for all $m \in \mathbb{R}$, and conversely, if $x \in \bigcap_{n>m} \mathcal{A}_\rho^n$, then $\rho(x) \leq n$ for all $n > m$ implying $\rho(x) \leq m$ and therefore $x \in \mathcal{A}_\rho^m$, showing the right-continuity. And so, $\mathcal{A}_\rho$ is a risk acceptance family.

Step 2. Let $\mathcal{A} = (\mathcal{A}^m)_{m \in \mathbb{R}}$ be a risk acceptance family and let $\rho_\mathcal{A}$ be the function defined as

$$\rho_\mathcal{A}(x) := \inf\{m \in \mathbb{R} : x \in \mathcal{A}^m\}, \quad x \in \mathcal{X}.$$

As for the monotonicity, consider $x \rhd y$ and fix[52] $m \in \mathbb{R}$ for which $y \in \mathcal{A}^m$. The monotonicity of $\mathcal{A}$ yields $x \in \mathcal{A}^m$. Hence, $m \geq \rho_\mathcal{A}(x)$ and therefore $\rho_\mathcal{A}(y) \geq \rho_\mathcal{A}(x)$. As for the quasiconvexity, let $\lambda \in]0,1[$, $x, y \in \mathcal{X}$ with $\rho_\mathcal{A}(x) \geq \rho_\mathcal{A}(y)$, and fix[53] $m \in \mathbb{R}$ for which $x \in \mathcal{A}^m$. The monotonicity implies that $y \in \mathcal{A}^m$ and by the convexity of $\mathcal{A}^m$ it follows $\lambda x + (1-\lambda) y \in \mathcal{A}^m$. This implies that $\rho_\mathcal{A}(\lambda x + (1-\lambda) y) \leq m$ and therefore

$$\rho_\mathcal{A}(\lambda x + (1-\lambda) y) \leq \rho_\mathcal{A}(x) = \max\{\rho_\mathcal{A}(x), \rho_\mathcal{A}(y)\}.$$

Hence, $\rho_\mathcal{A}$ is a risk measure.

Step 3. Let ρ be a risk measure. In view of the first and second step, $\rho_{\mathcal{A}_\rho}$ is also a risk measure. If $x \in \mathcal{X}$ is such that $\rho(x) = +\infty$, then it is unacceptable at any level of risk for $\mathcal{A}_\rho$, and therefore $\rho_{\mathcal{A}_\rho}(x) = +\infty$. The same argumentation holds for those $x \in \mathcal{X}$ satisfying $\rho(x) = -\infty$. If $\rho(x) \in \mathbb{R}$, then $x \in \mathcal{A}_\rho^{\rho(x)}$, hence $\rho_{\mathcal{A}_\rho}(x) \leq \rho(x)$. On the other hand, $x \notin \mathcal{A}_\rho^n$ for all $n < \rho(x)$, henceforth $\rho_{\mathcal{A}_\rho}(x) \geq \rho(x)$ and so $\rho = \rho_{\mathcal{A}_\rho}$.

Let $\mathcal{A}$ be a risk acceptance family. Due to the first and second step, $\mathcal{A}_{\rho_\mathcal{A}}$ is also a risk acceptance family. If $x \in \mathcal{A}^m$ for some $m \in \mathbb{R}$, it follows $\rho_\mathcal{A}(x) \leq m$ yielding $x \in \mathcal{A}_{\rho_\mathcal{A}}^m$. Conversely, $x \in \mathcal{A}_{\rho_\mathcal{A}}^m$ implies $\rho_\mathcal{A}(x) \leq m$, which in view of (1.7) yields $x \in \mathcal{A}^n$ for all $n > m$. The right-continuity of $\mathcal{A}$ implies $x \in \bigcap_{n>m} \mathcal{A}^n = \mathcal{A}^m$, and so $\mathcal{A} = \mathcal{A}_{\rho_\mathcal{A}}$.

C.2 Proof of Proposition 1.13: *(i):* Suppose that ρ is convex. Take $\lambda \in]0,1[$ and some reals m, m'. Any element of $\lambda \mathcal{A}^m + (1-\lambda) \mathcal{A}^{m'}$ can be written as $\lambda x + (1-\lambda) y$ for $x \in \mathcal{A}^m$ and $y \in \mathcal{A}^{m'}$. The convexity implies

$$\rho(\lambda x + (1-\lambda) y) \leq \lambda \rho(x) + (1-\lambda) \rho(y) \leq \lambda m + (1-\lambda) m',$$

showing that, $\lambda x + (1-\lambda) y \in \mathcal{A}^{\lambda m + (1-\lambda) m'}$. Conversely, by Theorem 1.7, for $m, m' \in \mathbb{R}$ such that $x \in \mathcal{A}^m$ and $y \in \mathcal{A}^{m'}$, it follows

$$
\begin{aligned}
\rho(\lambda x + (1-\lambda) y) &= \inf\{n \in \mathbb{R} : \lambda x + (1-\lambda) y \in \mathcal{A}^n\} \\
&= \inf_{n,n' \in \mathbb{R}}\{\lambda n + (1-\lambda) n' : \lambda x + (1-\lambda) y \in \mathcal{A}^{\lambda n + (1-\lambda) n'}\} \\
&\leq \inf_{n,n' \in \mathbb{R}}\{\lambda n + (1-\lambda) n' : \lambda x + (1-\lambda) y \in \lambda \mathcal{A}^n + (1-\lambda) \mathcal{A}^{n'}\} \leq \lambda m + (1-\lambda) m',
\end{aligned}
$$

showing that $\rho(\lambda x + (1-\lambda) y) \leq \lambda \rho(x) + (1-\lambda) \rho(y)$. The cases $\rho(x) = +\infty$ or $\rho(y) = +\infty$ are obvious.

(ii): Suppose that ρ is positive homogeneous. Fix $\lambda > 0$ and some $m \in \mathbb{R}$. Then, $x \in \mathcal{A}^{\lambda m}$ if and only if $\rho(x) \leq \lambda m$ if and only if $\rho(x/\lambda) \leq m$ if and only if $x \in \lambda \mathcal{A}^m$, that is, $\lambda \mathcal{A}^m = \mathcal{A}^{\lambda m}$. Conversely, Theorem 1.7

---

[52]The case where there is no such m is trivial as $\rho_\mathcal{A}(y) = +\infty$.

[53]Here again, the case where there is no such m is obvious.

yields

$$\rho\left(\lambda x\right) = \inf\left\{m \in \mathbb{R} : \lambda x \in \mathcal{A}^m\right\} = \inf\left\{m \in \mathbb{R} : x \in \mathcal{A}^{m/\lambda}\right\} = \lambda \inf\left\{m' \in \mathbb{R} : x \in \mathcal{A}^{m'}\right\} = \lambda\rho\left(x\right).$$

For the related risk order, since $\mathcal{L}\left(x\right) = \mathcal{A}^{\rho(x)}$, it follows

$$\lambda\mathcal{L}\left(x\right) = \lambda\mathcal{A}^{\rho(x)} = \mathcal{A}^{\lambda\rho(x)} = \mathcal{A}^{\rho(\lambda x)} = \mathcal{L}\left(\lambda x\right).$$

(iii): It is straightforward that any risk measure corresponding to a risk order satisfying $x \sim \lambda x$ for any $\lambda > 0$ is scaling invariant. As for the following assertions, the proof is analogous to the proof of *(ii)*.

C.3 Proof of Proposition 1.17: The cash additivity of a risk measure ρ corresponding to $\preccurlyeq$ obviously implies the properties (i) and (ii).

Conversely, conditions (i) and (ii) implies that the mapping

$$\rho(x) := \begin{cases} -\infty & \text{if } x \preccurlyeq y \text{ for any } y \in \mathcal{X} \\ -m & \text{if } y \prec x \prec z \text{ for some } y, z \in \mathcal{X} \text{ and } x \sim m\pi \\ +\infty & \text{if } y \preccurlyeq x \text{ for any } y \in \mathcal{X} \end{cases}$$

defines a cash additive risk measure corresponding to $\preccurlyeq$.

C.4 Proof of Proposition 1.18: Let ρ be a cash additive risk measure and fix some $m \in \mathbb{R}$. The respective risk acceptance family $\mathcal{A}$ satisfies

$$\mathcal{A}^m = \{x \in \mathcal{X} : \rho\left(x\right) \le m\} = \{x \in \mathcal{X} : \rho\left(x + m\pi\right) \le 0\} = \mathcal{A}^0 - m\pi,$$

and therefore fulfills the condition (1.12).

Conversely, let $\mathcal{A}$ be risk acceptance family satisfying relation (1.12). The cash additivity for the related risk measure ρ follows from Theorem 1.7 since

$$\rho\left(x + m\pi\right) = \inf\left\{m' \in \mathbb{R} : x \in \mathcal{A}^{m'}\right\} = \inf\left\{m' \in \mathbb{R} : x + \left(m + m'\right)\pi \in \mathcal{A}^0\right\} = \rho(x) - m$$

for any $x \in \mathcal{X}$ and $m \in \mathbb{R}$.

As for the convexity, let ρ be a cash additive risk measure. Proposition 1.18 implies that its risk acceptance family $\mathcal{A}$ fulfills the relation (1.12). Hence, for any $m, m' \in \mathbb{R}$ and $\lambda \in]0, 1[$ follows

$$\lambda\mathcal{A}^m + (1 - \lambda)\mathcal{A}^{m'} = \lambda\mathcal{A}^0 - \lambda m\pi + (1 - \lambda)\mathcal{A}^0 - (1 - \lambda)m'\pi$$
$$= \mathcal{A}^0 - \left(\lambda m + (1 - \lambda)m'\right)\pi = \mathcal{A}^{\lambda m + (1 - \lambda)m'}.$$

And so, by Proposition 1.13, ρ is convex.

C.5 Proof of Proposition 1.21: Let ρ be a cash subadditive risk measure with corresponding risk acceptance family $\mathcal{A}$. For any $m > 0$, $n \in \mathbb{R}$, and $x + m\pi \in \mathcal{A}^n$ follows $n \ge \rho\left(x + m\pi\right) \ge \rho(x) - m$, showing that $x \in \mathcal{A}^{m+n}$. Hence, $\mathcal{A}^n - m\pi \subseteq \mathcal{A}^{m+n}$.

Conversely, consider some risk acceptance family $\mathcal{A}$ fulfilling the relation (1.17) and with corresponding risk measure ρ. Theorem 1.7 yields for any $m > 0$ that

$$\rho\left(x + m\pi\right) = \inf\{n \in \mathbb{R} : x + m\pi \in \mathcal{A}^n\} \ge \inf\left\{n \in \mathbb{R} : x \in \mathcal{A}^{n+m}\right\} = \rho\left(x\right) - m,$$

showing that ρ is a cash subadditive risk measure.

C.6 Proof of Theorem 2.6: Throughout this subsection we assume the setup of Section 2.

Definition C.1. By $\mathcal{P}^{\min}$, we denote the set of *minimal penalty functions*, consisting of those mappings $\alpha : \mathcal{K}^\circ \times \mathbb{R} \to [-\infty, +\infty]$, which are nondecreasing and left-continuous in the second argument and such that:

    (*a*) α is convex in the first argument,

    (*b*) α is positive homogeneous in the first argument,

    (*c*) if there exists $x^* \in \mathcal{K}^\circ$ such that $\alpha(x^*, m) = -\infty$, then $\alpha(\cdot, m) \equiv -\infty$,

    (*d*) α is lower semicontinuous in the first argument.

We need two lemmata the first of which states a one-to-one relation between $\mathcal{P}^{\min}$ and $\mathcal{R}^{\max}$.

Lemma C.2. *The left inverse of any function $\alpha \in \mathcal{P}^{\min}$ is in $\mathcal{R}^{\max}$, that is*

$$\alpha^{(-1,l)}(x^*, s) := \sup\{m \in \mathbb{R} : \alpha(x^*, m) < s\} \in \mathcal{R}^{\max}. \tag{C.1}$$

The left inverse of any function $R \in \mathcal{R}^{\max}$ is in $\mathcal{P}^{\min}$, that is

$$R^{(-1,l)}(x^*, m) := \sup\{s \in \mathbb{R} : R(x^*, s) < m\} \in \mathcal{P}^{\min}. \tag{C.2}$$

Moreover, $(\alpha^{(-1,l)})^{(-1,l)} = \alpha$, as well as $(R^{(-1,l)})^{(-1,l)} = R$ for any $\alpha \in \mathcal{P}^{\min}$ and $R \in \mathcal{R}^{\max}$.

Proof. Note that both minimal penalty functions and maximal risk functions map $\mathcal{K}^\circ \times \mathbb{R}$ to $[-\infty, +\infty]$, and are nondecreasing left-continuous in the second argument. In the following, α is such a mapping from $\mathcal{K}^\circ \times \mathbb{R}$ to $[-\infty, +\infty]$. By Proposition B.2, its left inverse denoted by R is again a left-continuous nondecreasing function and in that case holds $\alpha = R^{(-1,l)} = (\alpha^{(-1,l)})^{(-1,l)}$, $R = \alpha^{(-1,l)} = (R^{(-1,l)})^{(-1,l)}$ and $R^+ = \alpha^{(-1,r)}$. Relation (B.5) in Proposition B.2 further implies that

$$R^+(x^*, s) \geq m \quad \text{if and only if} \quad s \geq \alpha(x^*, m), \tag{C.3}$$

for all $m, s \in \mathbb{R}$, and $x^* \in \mathcal{K}^\circ$.

We now show that $\alpha = R^{(-1,l)}$ is in $\mathcal{P}^{\min}$ if and only if $R = \alpha^{(-1,l)}$ is in $\mathcal{R}^{\max}$.

- Equivalence between condition (*a*) for $\mathcal{P}^{\min}$ and condition (*i*) for $\mathcal{R}^{\max}$. Firstly, since R is the left-continuous version of R^+ which itself is the right-continuous version of R holds by definition

$$\{(x^*, s) \in \mathcal{K}^\circ \times \mathbb{R} : R^+(x^*, s) \geq m\} = \bigcap_{\delta > 0} \{(x^*, s) \in \mathcal{K}^\circ \times \mathbb{R} : R(x^*, s + \delta) \geq m\},$$

$$\{(x^*, s) \in \mathcal{K}^\circ \times \mathbb{R} : R(x^*, s) \geq m\} = \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \{(x^*, s) \in \mathcal{K}^\circ \times \mathbb{R} : R^+(x^*, s - \delta) > m - \varepsilon\},$$

  showing the equivalence between the joint quasiconcavity of R and the joint quasiconcavity of R^+. Secondly, relation (C.3) yields

$$\{(x^*, s) \in \mathcal{K}^\circ \times \mathbb{R} : R^+(x^*, s) \geq m\} = \{(x^*, s) \in \mathcal{K}^\circ \times \mathbb{R} : s \geq \alpha(x^*, m)\} = \mathrm{epi}\,(\alpha(\cdot, m)),$$

  for any $m \in \mathbb{R}$. Finally, a function is convex if and only its epigraph is convex.

- Equivalence between condition (*b*) for $\mathcal{P}^{\min}$ and condition (*ii*) for $\mathcal{R}^{\max}$. If α is positive homogeneous in the first argument, then for any $\lambda > 0$ holds

$$R(\lambda x^*, s) = \sup\{m \in \mathbb{R} : \alpha(\lambda x^*, m) < s\} = \sup\{m \in \mathbb{R} : \alpha(x^*, m) < s/\lambda\} = R(x^*, s/\lambda).$$

  Conversely, under the assumption that $R(\lambda x^*, s) = R(x^*, s/\lambda)$ for all $\lambda > 0$, it follows

$$\alpha(\lambda x^*, m) = \sup\{s \in \mathbb{R} : R(\lambda x^*, s) < m\} = \lambda \sup\{s/\lambda \in \mathbb{R} : R(x^*, s/\lambda) < m\} = \lambda\alpha(x^*, m).$$

- Equivalence between condition (*c*) for $\mathcal{P}^{\min}$ and condition (*iii*) for $\mathcal{R}^{\max}$. Define

$$C := \{(x^*, m) \in \mathcal{K}^\circ \times \mathbb{R} : \alpha(x^*, m) = -\infty\},$$

$$D := \left\{(x^*, m) \in \mathcal{K}^\circ \times \mathbb{R} : \lim_{s \to -\infty} R(x^*, s) \geq m\right\}.$$

It is clear that if $\alpha \in \mathcal{P}^{\min}$, condition (*c*) for $\mathcal{P}^{\min}$ is equivalent to $C = \mathcal{K}^\circ \times J$ for the interval $J =]-\infty, c_0]$, where $c_0 \in [-\infty, +\infty]$. On the other hand, condition (*iii*) for $\mathcal{R}^{\max}$ holds if and only if $D = \mathcal{K}^\circ \times J$ for J as before. Indeed, $D = \mathcal{K}^\circ \times \emptyset$ if and only if $\lim_{s \to -\infty} R(x^*, s) = -\infty$ for any $x^* \in \mathcal{K}^\circ$. Further, $D = \mathcal{K}^\circ \times \mathbb{R}$ if and only if $R \equiv +\infty$. Finally, $D = \mathcal{K}^\circ \times]-\infty, c_0]$ for $c_0 \in \mathbb{R}$ if and only if $\lim_{s \to -\infty} R(x^*, s) = c_0$ for all $x^* \in \mathcal{K}^\circ$.

It remains to show that $C = D$. Indeed, relation (C.3) states that

$$C = \{(x^*, m) \in \mathcal{K}^\circ \times \mathbb{R} : \alpha(x^*, m) \leq s \text{ for all } s \in \mathbb{R}\}$$
$$= \{(x^*, m) \in \mathcal{K}^\circ \times \mathbb{R} : m \leq R^+(x^*, s) \text{ for all } s \in \mathbb{R}\}$$
$$= \{(x^*, m) \in \mathcal{K}^\circ \times \mathbb{R} : m \leq R(x^*, s) \text{ for all } s \in \mathbb{R}\} = D.$$

- Equivalence between condition (*d*) for $\mathcal{P}^{\min}$ and condition (*iv*) for $\mathcal{R}^{\max}$. Again by relation (C.3) holds

$$\{x^* \in \mathcal{K}^\circ : R^+(x^*, s) \geq m\} = \{x^* \in \mathcal{K}^\circ : s \geq \alpha(x^*, m)\}$$

for any $m, s \in \mathbb{R}$. This states the equivalence between the lower semicontinuity of α and the upper semicontinuity of R^+. $\qquad\square$

Let $\mathcal{P}_0^{\min}$ denote the set of positive homogeneous, lower semicontinuous and convex functions $\alpha : \mathcal{K}^\circ \to [-\infty, +\infty]$ such that if there exists $x^* \in \mathcal{K}^\circ$ with $\alpha(x^*) = -\infty$, then $\alpha \equiv -\infty$. In particular, if $\alpha \in \mathcal{P}^{\min}$ then $\alpha(\cdot, m) \in \mathcal{P}_0^{\min}$ for any $m \in \mathbb{R}$.

Lemma C.3. *Let $\mathcal{A} \subseteq \mathcal{X}$ be a $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed, convex and monotone[54] set. Then, there exists a unique $\alpha \in \mathcal{P}_0^{\min}$ such that*

$$x \in \mathcal{A} \quad \text{if and only if} \quad \langle x^*, -x \rangle \leq \alpha(x^*) \text{ for all } x^* \in \mathcal{K}^\circ. \tag{C.4}$$

In this case, α is given as the support function of $-\mathcal{A}$, that is, the minimal[55] penalty function

$$\alpha(x^*) = \alpha_{\min}(x^*) := \sup_{x \in \mathcal{A}} \langle x^*, -x \rangle, \quad x^* \in \mathcal{K}^\circ. \tag{C.5}$$

If in addition $\mathcal{K}$ is regular then for any fixed $\pi \in \tilde{\mathcal{K}}$ holds

$$x \in \mathcal{A} \quad \text{if and only if} \quad \langle x^*, -x \rangle \leq \alpha(x^*) \text{ for all } x^* \in \mathcal{K}_\pi^\circ, \tag{C.6}$$

and α is unique in the set of all lower semicontinuous convex functions from $\mathcal{K}_\pi^\circ$ to $[-\infty, +\infty]$ such that if there exists $x^* \in \mathcal{K}_\pi^\circ$ with $\alpha(x^*) = -\infty$, then $\alpha \equiv -\infty$.

Proof. Let $\alpha_{\min}$ denote the support function of $-\mathcal{A}$ given by relation (C.5). By definition, $\alpha_{\min} \in \mathcal{P}_0^{\min}$. We next show that $\alpha_{\min}$ fulfills relation (C.4). The cases $\mathcal{A} = \emptyset$ and $\mathcal{A} = \mathcal{X}$ are obvious. If $\mathcal{A} \neq \emptyset$, the implication

$$x \in \mathcal{A} \quad \text{implies} \quad \langle x^*, -x \rangle \leq \sup_{y \in \mathcal{A}} \langle x^*, -y \rangle = \alpha_{\min}(x^*), \quad \text{for all } x^* \in \mathcal{K}^\circ, \tag{C.7}$$

---

[54]That is $y \rhd x$ with $x \in \mathcal{A}$ implies $y \in \mathcal{A}$.
[55]The minimality of the penalty function follows from the arguments given in [30, Theorem 4.15].

is straightforward. Conversely, for any $x \in \mathcal{X} \setminus \mathcal{A}$, the hyperplane separation theorem yields

$$\langle x_0^*, -x \rangle > \sup_{y \in \mathcal{A}} \langle x_0^*, -y \rangle \tag{C.8}$$

for some $x_0^* \in \mathcal{X}^*$. By the monotonicity of $\mathcal{A}$ holds $\langle x_0^*, -x \rangle > \langle x_0^*, -y \rangle + \langle x_0^*, -k \rangle$ for some $y \in \mathcal{A}$ and all $k \in \mathcal{K}$. Hence, $0 \geq \langle x_0^*, -k \rangle$ for all $k \in \mathcal{K}$, implying that $x_0^* \in \mathcal{K}^\circ$ and therefore, the right-hand side of (C.8) is equal to $\alpha_{\min}(x_0^*)$ showing the reverse implication in (C.4).

As for the uniqueness, suppose there exist $\alpha_1, \alpha_2 \in \mathcal{P}_0^{\min}$ which represent $\mathcal{A}$ in the sense of (C.4). In case that α_1 is identically $+\infty$ or $-\infty$, the same obviously holds for α_2 and vice versa. By definition of $\mathcal{P}_0^{\min}$, it remains to show the case where both α_1 and α_2 are proper. Define $\tilde{\alpha}_i = \alpha_i$ on $\mathcal{K}^\circ$ and $\tilde{\alpha}_i = +\infty$ on $\mathcal{K}^{\circ c}$ which remains proper, convex and lower semicontinuous. For the conjugates $\tilde{\alpha}_i^*(x) = \sup_{x^* \in \mathcal{K}^\circ} \{\langle x^*, x \rangle - \alpha_i(x^*)\}$ which are positive homogeneous follow $\tilde{\alpha}_i^*(x) = 0$ if and only if $-x \in \mathcal{A}$. Thus, $\tilde{\alpha}_1^* = \tilde{\alpha}_2^*$ and the Fenchel-Moreau theorem yields $\tilde{\alpha}_1 = (\tilde{\alpha}_1^*)^* = (\tilde{\alpha}_2^*)^* = \tilde{\alpha}_2$, that is, $\alpha_1 = \alpha_2$ on $\mathcal{K}^\circ$.

Finally, in case that $\pi \in \hat{\mathcal{K}} \neq \emptyset$, it follows that $\langle x^*, \pi \rangle > 0$ for any $x^* \in \mathcal{K}^\circ \setminus \{0\}$ so that $x^*/\langle x^*, \pi \rangle \in \mathcal{K}_\pi^\circ$. Hence, $\mathcal{K}^\circ = \mathbb{R}_+ \mathcal{K}_\pi^\circ$ and (C.4) is equivalent to

$$x \in \mathcal{A} \quad \text{if and only if} \quad \left\langle \frac{x^*}{\langle x^*, \pi \rangle}, -x \right\rangle \leq \alpha \left(\frac{x^*}{\langle x^*, \pi \rangle} \right) \quad \text{for all } x^* \in \mathcal{K}^\circ \setminus \{0\}. \qquad \square$$

Proof (Theorem 2.6). **Step 1.** Let ρ be a lower semicontinuous risk measure. Theorem 1.7 yields

$$\rho(x) = \inf \{m \in \mathbb{R} : x \in \mathcal{A}^m\}, \quad x \in \mathcal{X}. \tag{C.9}$$

Since any $\mathcal{A}^m$ is $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed, convex and monotone, it follows from Lemma C.3 that

$$x \in \mathcal{A}^m \quad \text{if and only if} \quad \langle x^*, -x \rangle \leq \alpha_{\min}(x^*, m) \text{ for all } x^* \in \mathcal{K}^\circ, \tag{C.10}$$

whereby $\alpha_{\min}(\cdot, m)$ is the support function of $-\mathcal{A}^m$ as given by relation (C.5). Combining (C.9) and (C.10) yields

$$\begin{aligned}
\rho(x) &= \inf \{m \in \mathbb{R} : \langle x^*, -x \rangle \leq \alpha_{\min}(x^*, m) \text{ for all } x^* \in \mathcal{K}^\circ\} \\
&= \inf \{m \in \mathbb{R} : \langle x^*, -x \rangle \leq \alpha_{\min}^-(x^*, m) \text{ for all } x^* \in \mathcal{K}^\circ\},
\end{aligned} \tag{C.11}$$

for the left-continuous version $\alpha_{\min}^-$ of $\alpha_{\min}$. The goal is to show that

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} \inf_{m \in \mathbb{R}} \{m : \langle x^*, -x \rangle \leq \alpha_{\min}^-(x^*, m)\}. \tag{C.12}$$

To begin with, equation (C.11) implies:

$$\rho(x) \geq \sup_{x^* \in \mathcal{K}^\circ} \inf_{m \in \mathbb{R}} \{m : \langle x^*, -x \rangle \leq \alpha_{\min}^-(x^*, m)\}.$$

As for the reverse inequality, suppose that $\rho(x) > -\infty$, otherwise (C.12) is trivial, and fix $m_0 < \rho(x)$. Define $C = \{y \in \mathcal{X} : \rho(y) \leq m_0\}$, which is $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed, convex, and such that $x \notin C$. By the hyperplane separation theorem, there exists $x_0^* \in \mathcal{X}^* \setminus \{0\}$ such that

$$\langle x_0^*, x \rangle < \inf_{y \in C} \langle x_0^*, y \rangle. \tag{C.13}$$

By monotonicity of ρ we have $C = C + \mathcal{K}$, hence (C.13) yields $\langle x_0^*, x \rangle < \langle x_0^*, y \rangle + \langle x_0^*, k \rangle$ for all $k \in \mathcal{K}$ and $y \in C$. It follows that[56] $x_0^* \in \mathcal{K}^\circ \setminus \{0\}$. Since $\mathcal{A}^m \subseteq C$ for all $m \leq m_0$, (C.13) yields

$$\langle x_0^*, -x \rangle - \alpha_{\min}(x_0^*, m) \geq \langle x_0^*, -x \rangle - \sup_{y \in C} \langle x_0^*, -y \rangle > 0. \tag{C.14}$$

---

[56] In case that $C = \emptyset$, x_0^* can arbitrarily be chosen in $\mathcal{K}^\circ \setminus \{0\}$.

Hence, since $m \mapsto \alpha_{\min}(x_0^*, m)$ is nondecreasing,

$$m_0 \leq \sup_{x^* \in \mathcal{K}^\circ} \inf_{m \in \mathbb{R}} \{m : \langle x^*, -x \rangle \leq \alpha_{\min}(x^*, m)\} = \sup_{x^* \in \mathcal{K}^\circ} \inf_{m \in \mathbb{R}} \{m : \langle x^*, -x \rangle \leq \alpha_{\min}^-(x^*, m)\}. \quad (C.15)$$

Since the last relation holds for any $m_0 < \rho(x)$ we derive

$$\rho(x) \leq \sup_{x^* \in \mathcal{K}^\circ} \inf_{m \in \mathbb{R}} \{m : \langle x^*, -x \rangle \leq \alpha_{\min}^-(x^*, m)\},$$

and (C.12) is established.

Step 2. Since

$$\alpha_{\min}^-(x^*, m) = \sup_{m' < m} \sup_{x \in \mathcal{A}^{m'}} \langle x^*, -x \rangle = \sup_{x \in \mathcal{A}^{m-}} \langle x^*, -x \rangle,$$

where $\mathcal{A}^{m-} = \bigcup_{m' < m} \mathcal{A}^{m'}$ is closed, a direct inspection shows that $\alpha_{\min}^- \in \mathcal{P}^{\min}$. According to Lemma C.2, the left inverse of $\alpha_{\min}^-$, denoted by R is a maximal risk function, that is $R \in \mathcal{R}^{\max}$ and therefore relation (C.12) yields

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} R(x^*, \langle x^*, -x \rangle), \quad x \in \mathcal{X}. \quad (C.16)$$

As for the uniqueness, according to Lemma C.2 it is sufficient to show the uniqueness of $\alpha_{\min}^-$ in (C.12) as $\alpha_{\min}^- \in \mathcal{P}^{\min}$. Consider $\alpha_1, \alpha_2 \in \mathcal{P}^{\min}$ satisfying

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} \inf_{m \in \mathbb{R}} \{m : \langle x^*, -x \rangle \leq \alpha_i(x^*, m)\}, \quad x \in \mathcal{X},$$

for $i = 1, 2$. For any $m \in \mathbb{R}$ holds

$$\begin{aligned}
\{x \in \mathcal{X} : \rho(x) < m\} &= \bigcup_{m' < m} \{x \in \mathcal{X} : \sup_{x^* \in \mathcal{K}^\circ} \inf_{n \in \mathbb{R}} \{n : \langle x^*, -x \rangle \leq \alpha_i(x^*, n)\} \leq m'\} \\
&= \bigcup_{m' < m} \{x \in \mathcal{X} : \inf_{n \in \mathbb{R}} \{n : \langle x^*, -x \rangle \leq \alpha_i(x^*, n)\} \leq m' \text{ for all } x^* \in \mathcal{K}^\circ\} \\
&= \bigcup_{m' < m} \{x \in \mathcal{X} : \langle x^*, -x \rangle \leq \alpha_i(x^*, m' + (m - m')/2) \text{ for all } x^* \in \mathcal{K}^\circ\} \\
&= \bigcup_{m' < m} \{x \in \mathcal{X} : \langle x^*, -x \rangle \leq \alpha_i(x^*, m') \text{ for all } x^* \in \mathcal{K}^\circ\} = \bigcup_{m' < m} \mathcal{A}_i^{m'}, \quad (C.17)
\end{aligned}$$

for the $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed convex sets $\mathcal{A}_i^{m'} := \{x : \langle x^*, -x \rangle \leq \alpha_i(x^*, m') \text{ for all } x^* \in \mathcal{K}^\circ\}$. The uniqueness result in Lemma C.3 yields

$$\alpha_i(x^*, m') = \sup_{x \in \mathcal{A}_i^{m'}} \langle x^*, -x \rangle. \quad (C.18)$$

Thus, from relations (C.17), (C.18) and the left-continuity of $\alpha_i(x^*, \cdot)$ it follows

$$\alpha_i(x^*, m) = \sup_{m' < m} \alpha_i(x^*, n) = \sup_{m' < m} \sup_{x \in \mathcal{A}_i^{m'}} \langle x^*, -x \rangle = \sup_{\bigcup_{m' < m} \mathcal{A}_i^{m'}} \langle x^*, -x \rangle = \sup_{\{x : \rho(x) < m\}} \langle x^*, -x \rangle,$$

and therefore $\alpha_1 = \alpha_2$.

Step 3. Conversely, let $\rho(x) := \sup_{x^* \in \mathcal{K}^\circ} R(x^*, \langle x^*, -x \rangle)$ for a risk function $R \in \mathcal{R}$. Since $s \mapsto R(x^*, s)$ is nondecreasing, it follows that ρ is monotone. Further, $s \mapsto R(x^*, s)$ is left-continuous, nondecreasing and $x \mapsto \langle x^*, -x \rangle$ is linear and continuous for all $x^* \in \mathcal{K}^\circ$. In view of relation (C.3) it holds

$$\{x \in \mathcal{X} : R(x^*, \langle x^*, -x \rangle) \leq m\} = \{x \in \mathcal{X} : \langle x^*, -x \rangle \leq R^{(-1,r)}(x^*, m)\},$$

and therefore $x \mapsto R\left(x^*, \langle x^*, -x \rangle\right)$ is a lower semicontinuous quasiconvex function. This implies that the level sets

$$
\begin{aligned}
\{x \in \mathcal{X} : \rho(x) \leq m\} &= \left\{ x \in \mathcal{X} : \sup_{x^* \in \mathcal{K}^\circ} R\left(x^*, \langle x^*, -x \rangle\right) \leq m \right\} \\
&= \bigcap_{x^* \in \mathcal{K}^\circ} \{x \in \mathcal{X} : R\left(x^*, \langle x^*, -x \rangle\right) \leq m\}
\end{aligned}
$$

are closed and convex for all $m \in \mathbb{R}$. Hence, ρ is a lower semicontinuous risk measure. $\qquad\square$

C.7 Proof of Theorem 2.7: Let ρ be a lower semicontinuous risk measure. By Theorem 2.6 there exists a unique $R \in \mathcal{R}^{\max}$ whose restriction to $\mathcal{K}_\pi^\circ \times \mathbb{R}$ is in $\mathcal{R}_\pi^{\max}$ and such that

$$
\rho(x) = \sup_{x^* \in \mathcal{K}_\pi^\circ, \, \lambda > 0} R\left(\lambda x^*, \langle \lambda x^*, -x \rangle\right) = \sup_{x^* \in \mathcal{K}_\pi^\circ} R\left(x^*, \langle x^*, -x \rangle\right).
$$

Due to the condition (*ii*) for $\mathcal{R}^{\max}$, there is a one-to-one relation between $\mathcal{R}^{\max}$ and $\mathcal{R}_\pi^{\max}$, from which the uniqueness follows.

C.8 Proof of Proposition 2.9: Let $R, \tilde{R}$ be two risk functions such that

$$
\rho\left(x\right) = \sup_{x^* \in \mathcal{K}^\circ} R\left(x^*, \langle x^*, -x \rangle\right) = \sup_{x^* \in \mathcal{K}^\circ} \tilde{R}\left(x^*, \langle x^*, -x \rangle\right),
$$

where $R \in \mathcal{R}^{\max}$. According to Theorem 2.6, R is the left inverse of $\alpha_{\min}$. Further, $\tilde{R}\left(x^*, \langle x^*, -x \rangle\right) \leq m$ for all $x^* \in \mathcal{K}^\circ$, $m \in \mathbb{R}$ and $x \in \mathcal{A}^m$. Hence, relation (B.4) yields $\langle x^*, -x \rangle \leq \tilde{R}^{(-1,r)}\left(x^*, m\right)$ for all $x^* \in \mathcal{K}^\circ$, $m \in \mathbb{R}$ and $x \in \mathcal{A}^m$. Due to Lemma C.3, $\alpha_{\min}$ is the smallest function having this property, that is, $\alpha_{\min}\left(x^*, m\right) \leq \tilde{R}^{(-1,r)}\left(x^*, m\right)$ for all $x^* \in \mathcal{X}^*$ and $m \in \mathbb{R}$. Since by Theorem 2.6 the maximal risk function R is the left inverse of $\alpha_{\min}$, it follows $\tilde{R}\left(x^*, s\right) \leq R\left(x^*, s\right)$ for all $x^* \in \mathcal{X}^*$ and $s \in \mathbb{R}$.

C.9 Technical examples referring to Remark 2.10:

Example C.4 (Importance of property (*iv*) in Definition 2.5). Let $\mathcal{X} = \mathbb{R}^2$, $\mathcal{K} = \mathbb{R}_+^2$, and $\pi = (1,1)$ in which case $\mathcal{K}_\pi^\circ = \{(p, 1-p) : p \in [0,1]\}$. As for the first example, consider the risk function $R(p, s) = 1_{\{s > p\}}$ which is in $\mathcal{R}^{\max}$. However, $R(\cdot, 1/2)$ is not upper semicontinuous, since $\{p \in [0,1] : R(p, 1/2) \geq 1/2\} = [0, 1/2)$ is not closed. In the second example, we show that maximal risk functions are in general not lower semicontinuous in the first argument. Indeed, within the setup of the previous example, we consider the maximal risk function $R(p, s) = 1_{\{p \geq 1/2\}}$ for which $R = R^+$, but $R(\cdot, s)$ is for any $s \in \mathbb{R}$ not lower semicontinuous. $\qquad\diamond$

Example C.5 (Importance of the regularity assumption for Theorem 2.7). Indeed, let $\mathcal{X} = \mathbb{R}$, $\mathcal{K} = \{0\}$ so that $\mathcal{K}^\circ = \mathcal{X}^* = \mathbb{R}$ and consider the lower semicontinuous quasiconvex function $\rho(x) := x^2$, which is monotone with respect to the non-regular preorder $\mathcal{K} = \{0\}$. However, there does not exist any $\pi \in \mathbb{R} \setminus \{0\}$ such that

$$
\rho(x) = \sup_{x^* \in \mathcal{K}_\pi^\circ} R\left(x^*, -x^* \cdot x\right), \quad \text{for all } x \in \mathbb{R},
$$

as $\mathcal{K}_\pi^\circ = \{x^* \in \mathbb{R} : x^* \pi = 1\}$ reduces to the singleton $1/\pi$ and $\rho(x) = x^2$ is different from any function $x \mapsto R\left(1/\pi, -x/\pi\right)$ for some $R \in \mathcal{R}$, which by definition is either nondecreasing or nonincreasing depending on the sign of π. $\qquad\diamond$

C.10 Proof of Proposition 2.11: The proof is built on the respective properties of the acceptance family, which have been established in Propositions 1.13, 1.18, and 1.21.

In case that R is convex, positive homogeneous or scaling invariant in the second argument, it follows that ρ is convex, positive homogeneous or scaling invariant as the supremum of convex, positive homogeneous or scaling invariant functions is convex, positive homogeneous or scaling invariant, respectively.

Conversely, suppose that ρ is convex. By Proposition 1.13, for any $m, m' \in \mathbb{R}$, $\lambda \in\,]0, 1[$ and $x^* \in \mathcal{K}^\circ$ holds

$$\alpha_{\min}\left(x^*, \lambda m + (1 - \lambda)\, m'\right) = \sup_{x \in \mathcal{A}^{\lambda m + (1-\lambda)m'}} \langle x^*, -x \rangle \geq \sup_{x \in \lambda \mathcal{A}^m + (1-\lambda)\mathcal{A}^{m'}} \langle x^*, -x \rangle$$

$$= \lambda \sup_{x \in \mathcal{A}^m} \langle x^*, -x \rangle + (1 - \lambda) \sup_{x \in \mathcal{A}^{m'}} \langle x^*, -x \rangle = \lambda \alpha_{\min}\left(x^*, m\right) + (1 - \lambda)\, \alpha_{\min}\left(x^*, m'\right).$$

Hence, $m \mapsto \alpha_{\min}^+\left(x^*, m\right)$ is concave. The function $R\left(x^*, \cdot\right)$ as the left inverse of $\alpha_{\min}^+\left(x^*, \cdot\right)$ is therefore convex by Proposition B.2. Analoguously, if ρ is positive homogeneous or scaling invariant, it follows from Proposition 1.13 that $\alpha_{\min}(x^*, \lambda m) = \lambda \alpha_{\min}(x^*, m)$ or $\alpha_{\min}(x^*, \lambda m) = \alpha_{\min}(x^*, m)$ and thus R is positive homogeneous or scaling invariant in the second argument, respectively.

For the cash additive and cash subadditive case, the suffciency is obvious. Conversly, if ρ is cash additive, Proposition 1.18 implies $\alpha_{\min}\left(x^*, m\right) = \alpha_{\min}\left(x^*, 0\right) + m$ from which we deduce $R\left(x^*, s\right) = s - \alpha_{\min}\left(x^*, 0\right)$, and therefore $R\left(x^*, s + m\right) = R\left(x^*, s\right) + m$ for all $m, s \in \mathbb{R}$ and $x^* \in \mathcal{K}_\pi^\circ$. If ρ is cash subadditive, it follows from Proposition 1.18 that $\alpha_{\min}\left(x^*, n + m\right) \geq \alpha_{\min}\left(x^*, n\right) + m$ for all $x^* \in \mathcal{K}_\pi^\circ$, $n \in \mathbb{R}$ and $m > 0$. Hence, for any $x^* \in \mathcal{K}_\pi^\circ$, $s \in \mathbb{R}$ and $m > 0$ holds

$$R\left(x^*, s - m\right) = \sup_{n \in \mathbb{R}} \left\{n : \alpha_{\min}\left(x^*, n\right) < s - m\right\}$$

$$\geq \sup_{n \in \mathbb{R}} \left\{n : \alpha_{\min}\left(x^*, n + m\right) < s\right\} = R\left(x^*, s\right) - m.$$

C.11 Proof of Propositions 2.2 and 2.15: The sufficiency in both propositions is immediate. As for the necessity, the separability implies the existence of a risk measure ρ. By Debreu [24, 25]'s gap theorem, we can assume, up to a strictly increasing transformation, that $\mathrm{Im}\left(\rho\right)^c$ consists of intervals of either the form $[a, b]$ or $]a, b[$ for $a, b \in \mathbb{R}$. The lower semicontinuity and continuously extensible properties of $\preccurlyeq$ imply the same properties for ρ respectively. Indeed, let $\mathcal{A}$ be the corresponding risk acceptance family and $m \in \mathbb{R}$.

- If $m \in \mathrm{Im}\left(\rho\right)$ or m in a gap $]a, b[\, \subseteq \mathrm{Im}\left(\rho\right)^c$, then $\mathcal{A}^m = \mathcal{L}\left(x\right)$ for $\rho\left(x\right) = m$ or $\mathcal{A}^m = \mathcal{A}^a = \mathcal{L}\left(x\right)$ for $\rho\left(x\right) = a$. In both cases, $\mathcal{A}^m$ is closed if $\preccurlyeq$ is lower semicontinuous. For the same choice of x, the continuously extensible assumption on $\preccurlyeq$ yields $\overline{\mathcal{A}^m + \mathcal{K}} \cap \mathcal{X} = \overline{\mathcal{L}\left(x\right) + \mathcal{K}} \cap \mathcal{X} = \mathcal{L}\left(x\right) = \mathcal{A}^m$.

- If m is in a gap $[a, b] \subseteq \mathrm{Im}\left(\rho\right)^c$, the right-continuity of $\mathcal{A}$ implies the existence of a sequence of $\left(x_l\right)$ with $\rho\left(x_l\right) \searrow b$ such that $\mathcal{A}^m = \bigcap_{l \in \mathbb{N}} \mathcal{L}\left(x_l\right)$. Hence $\mathcal{A}^m$ is closed if $\preccurlyeq$ is lower semicontinuous. For the same sequence $\left(x_l\right)$, the continuously extensible assumption implies $\overline{\mathcal{A}^m + \mathcal{K}} \cap \mathcal{X} \subseteq \overline{\mathcal{L}\left(x_l\right) + \mathcal{K}} \cap \mathcal{X} = \mathcal{L}\left(x_l\right)$ for any $l \in \mathbb{N}$, and therefore

$$\mathcal{A}^m \subseteq \overline{\mathcal{A}^m + \mathcal{K}} \cap \mathcal{X} \subseteq \bigcap_{l \in \mathbb{N}} \overline{\mathcal{L}\left(x_l\right) + \mathcal{K}} \cap \mathcal{X} = \bigcap_{l \in \mathbb{N}} \mathcal{L}\left(x_l\right) = \mathcal{A}^m.$$

Let now $\hat{\rho} = h \circ \rho$ for a continuously extensible lower semicontinuous risk measure $\rho : \mathcal{X} \to [-\infty, \infty]$ and a lower semicontinuous increasing function $h : Im\left(\rho\right) \to \mathbb{R}$. By relation (B.4),

$$\hat{\mathcal{A}}^m = \left\{x \in \mathcal{X} : \hat{\rho}\left(x\right) = h \circ \rho\left(x\right) \leq m\right\} = \left\{x \in \mathcal{X} : \rho\left(x\right) \leq h^{(-1,r)}\left(m\right)\right\} = \mathcal{A}^{h^{(-1,r)}(m)},$$

and so the lower semicontinuity of ρ implies the lower semicontinuity of $\hat{\rho}$. Furthermore, the continuously extensible assumption on ρ yields

$$\overline{\hat{\mathcal{A}}^m + \mathcal{K}} \cap \mathcal{X} = \overline{\mathcal{A}^{h^{(-1,r)}(m)} + \mathcal{K}} \cap \mathcal{X} = \mathcal{A}^{h^{(-1,r)}(m)} = \hat{\mathcal{A}}^m.$$

C.12 Proof of Proposition 2.16: Fix $R : \mathcal{K}^\circ \times \mathbb{R} \to [-\infty, \infty]$ and define $\mathcal{H} := \left\{\tilde{R} \in \mathcal{R}^{\max} : \tilde{R} \geq R\right\}$. We have to show that the function $\bar{R} := \inf_{\tilde{R} \in \mathcal{H}} \tilde{R} \in \mathcal{R}^{\max}$. To this end, we first notice that $\bar{R}$ as an infimum of jointly

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quasiconcave functions is jointly quasiconcave. Secondly, for any $x^* \in \mathcal{K}^\circ$, $s \in \mathbb{R}$ and $\lambda > 0$ holds

$$\bar{R}(\lambda x^*, s) = \inf_{\tilde{R} \in \mathcal{H}} \tilde{R}(\lambda x^*, s) = \inf_{\tilde{R} \in \mathcal{H}} \tilde{R}(x^*, s/\lambda) = \bar{R}(x^*, s/\lambda).$$

Thirdly, $\bar{R}$ has a uniform asymptotic minimum as for any $x^*, y^* \in \mathcal{K}^\circ$

$$\lim_{s \to -\infty} \bar{R}(x^*, s) = \inf_{s \in \mathbb{R}} \inf_{\tilde{R} \in \mathcal{H}} \tilde{R}(x^*, s) = \inf_{\tilde{R} \in \mathcal{H}} \inf_{s \in \mathbb{R}} \tilde{R}(x^*, s) = \inf_{\tilde{R} \in \mathcal{H}} \inf_{s \in \mathbb{R}} \tilde{R}(y^*, s) = \lim_{s \to -\infty} \bar{R}(y^*, s).$$

Finally, $\bar{R}^+$ is upper semicontinuous in the second argument as for any $s, m \in \mathbb{R}$, the set

$$\left\{ x^* : \bar{R}^+(x^*, s) \geq m \right\} = \bigcap_{\tilde{R} \in \mathcal{H},\, t > s} \left\{ x^* : \tilde{R}(x^*, t) \geq m \right\} = \bigcap_{\tilde{R} \in \mathcal{H}} \left\{ x^* : \tilde{R}^+(x^*, s) \geq m \right\} \text{ is closed.}$$

C.13 Proof of Theorem 2.19: For the proof, we use other notions of closures. For a function $h : \mathcal{K}^\circ \times \mathbb{R} \to [-\infty, +\infty]$ and a function $g : \mathcal{V}^* \to [-\infty, +\infty]$,

- recall that the closure in $\mathcal{R}^{\max}$ of h denoted by $\mathrm{cl}_{\mathcal{R}^{\max}}(h)$ is given by

$$\mathrm{cl}_{\mathcal{R}^{\max}}(h)(x^*, s) := \inf \left\{ \tilde{R}(x^*, s) : \tilde{R} \geq h \text{ and } \tilde{R} \in \mathcal{R}^{\max} \right\}, \quad (x^*, s) \in \mathcal{K}^\circ \times \mathbb{R}.$$

- the closure in $\mathcal{P}^{\min}$ of h denoted by $\mathrm{cl}_{\mathcal{P}^{\min}}(h)$ is given by

$$\mathrm{cl}_{\mathcal{P}^{\min}}(h)(x^*, s) := \sup \left\{ \tilde{\alpha}(x^*, s) : \tilde{\alpha} \leq h \text{ and } \tilde{\alpha} \in \mathcal{P}^{\min} \right\}, \quad (x^*, s) \in \mathcal{K}^\circ \times \mathbb{R}.$$

- the convex closure of g denoted by $\mathrm{cl}(g) : \mathcal{V}^* \to [-\infty, +\infty]$ is either uniformly equal to $-\infty$ if $g(x^*) = -\infty$ for some $x^* \in \mathcal{V}^*$, or is the greatest lower semicontinuous convex function majorized by g.

A similar argumentation as for the proof of Proposition 2.16 shows that for $\mathrm{cl}_{\mathcal{P}^{\min}}(h) \in \mathcal{P}^{\min}$.

Lemma C.6. *Let $\alpha : \mathcal{K}^\circ \times \mathbb{R} \to [-\infty, \infty]$ be nondecreasing in the second argument, then*[57]

$$\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha) = [\mathrm{cl}(\alpha)]^- \quad \text{and} \quad \mathrm{cl}_{\mathcal{R}^{\max}}\left(\alpha^{(-1,l)}\right) = [\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha)]^{(-1,l)}.$$

Proof. Fix $\beta \in \mathcal{P}^{\min}$ with $[\mathrm{cl}(\alpha)]^- \leq \beta \leq \alpha$. Since $[\mathrm{cl}(\alpha)]^- \in \mathcal{P}^{\min}$ and $[\mathrm{cl}(\beta)]^- = [\beta]^- = \beta$ it follows $[\mathrm{cl}(\alpha)]^- \leq \beta \leq [\mathrm{cl}(\alpha)]^-$ and thus $\beta = [\mathrm{cl}(\alpha)]^-$, that is, $\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha) = [\mathrm{cl}(\alpha)]^-$. As for the second equality, by Lemma C.2 holds $\alpha^{(-1,l)} \leq [\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha)]^{(-1,l)} \in \mathcal{R}^{\max}$. Fix $\tilde{R} \in \mathcal{R}^{\max}$ satisfying $\alpha^{(-1,l)} \leq \tilde{R} \leq [\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha)]^{(-1,l)}$. Then, $\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha) \leq \tilde{R}^{(-1,l)} \leq \alpha$ and since by Lemma C.2 $\tilde{R}^{(-1,l)} \in \mathcal{P}^{\min}$, we deduce that $\tilde{R}^{(-1,l)} = \mathrm{cl}_{\mathcal{P}^{\min}}(\alpha)$, which in turn implies $\tilde{R} = [\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha)]^{(-1,l)}$. This shows that the $\mathrm{cl}_{\mathcal{R}^{\max}}$-closure of $\alpha^{(-1,l)}$ is $[\mathrm{cl}_{\mathcal{P}^{\min}}(\alpha)]^{(-1,l)}$. $\qquad\qquad \square$

Proof (of Theorem 2.19). Given a continuously extensible lower semicontinuous risk measure ρ with corresponding acceptance family $\mathcal{A}$, define the family $\hat{\mathcal{A}}$ by $\hat{\mathcal{A}}^m = \bigcap_{n>m} \overline{(\mathcal{A}^n + \mathcal{K})}$ for $m \in \mathbb{R}$. By means of relation (2.10), it is straightforward to check that $\hat{\mathcal{A}}$ is the smallest closed acceptance family in $\mathcal{V}$, for which $\hat{\mathcal{A}}^m \cap \mathcal{X} = \mathcal{A}^m$ for all $m \in \mathbb{R}$. Hence, the corresponding risk measure $\hat{\rho}$ on $\mathcal{V}$ is lower semicontinuous and coincides with ρ on $\mathcal{X}$. Furthermore, since $\hat{\mathcal{A}}$ is the smallest closed risk acceptance family containing $\mathcal{A}$, it follows that $\hat{\rho}$ is the unique maximal lower semicontinuous risk measure extension of ρ on $\mathcal{V}$. According to Theorem 2.6, denote by $\hat{R}$ the unique element in $\mathcal{R}^{\max}$, such that

$$\hat{\rho}(x) = \sup_{x^* \in \mathcal{K}^\circ} \hat{R}(x^*, \langle x^*, -x \rangle), \quad x \in \mathcal{V}.$$

We next show that any lower semicontinuous risk measure $\tilde{\rho}(x) = \sup_{x^* \in \mathcal{K}^\circ} \tilde{R}(x^*, \langle x^*, -x \rangle)$ on $\mathcal{V}$ with $\tilde{R} \in \mathcal{R}^{\max}$ is a maximal extension of the lower semicontinuous risk measure ρ on $\mathcal{X}$ if and only if $\tilde{R} \in \mathcal{R}_{\mathcal{X}}^{\max}$. Indeed,

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[57]Here, $\mathrm{cl}(\alpha)$ is the convex closure with respect to the first argument and $[\cdot]^-$ is the left-continuous version in the second argument.

since $\tilde{\mathcal{A}}^m \cap \mathcal{X} = \mathcal{A}^m$ and $\hat{\mathcal{A}}^m \subseteq \tilde{\mathcal{A}}^m$ for all $m \in \mathbb{R}$, it follows

$$\tilde{\rho} \text{ is a maximal extension} \quad \Longleftrightarrow \quad \tilde{\mathcal{A}} = \hat{\mathcal{A}} \quad \Longleftrightarrow \quad \tilde{\mathcal{A}}^m = \bigcap_{n>m} \overline{\left(\tilde{\mathcal{A}}^n \cap \mathcal{X}\right) + \mathcal{K}} \quad \text{for all } m \in \mathbb{R}.$$

Due to Lemma C.3 holds

$$\tilde{\mathcal{A}} = \hat{\mathcal{A}} \quad \Longleftrightarrow \quad \sup_{x \in \bigcap_{n>m} \overline{(\tilde{\mathcal{A}}^n \cap \mathcal{X}) + \mathcal{K}}} \langle x^*, -x \rangle = \sup_{x \in \tilde{\mathcal{A}}^m} \langle x^*, -x \rangle, \text{ for all } x^* \in \mathcal{K}^\circ \text{ and } m \in \mathbb{R}.$$

Denoting $\tilde{\alpha}_{\min}(x^*, m) := \sup_{x \in \tilde{\mathcal{A}}^m} \langle x^*, -x \rangle$ for $x^* \in \mathcal{V}^*$, we deduce from (A.3) that

$$\text{cl}\left(\tilde{\alpha}_{min} \square \delta_{\mathcal{X}}\right)(x^*, m) + \delta_{\mathcal{K}}(x^*) = \sup_{x \in \left(\tilde{\mathcal{A}}^m \cap \mathcal{X}\right) + \mathcal{K}} \langle x^*, -x \rangle \le \sup_{x \in \bigcap_{n>m} \overline{(\tilde{\mathcal{A}}^n \cap \mathcal{X}) + \mathcal{K}}} \langle x^*, -x \rangle$$

$$\le \inf_{n>m} \sup_{x \in \overline{(\tilde{\mathcal{A}}^n \cap \mathcal{X}) + \mathcal{K}}} \langle x^*, -x \rangle$$

$$= \left[\text{cl}\left(\tilde{\alpha}_{\min} \square \delta_{\mathcal{X}}\right)\right]^+ (x^*, m) + \delta_{\mathcal{K}}(x^*), \quad \text{for all } x^* \in \mathcal{K}^\circ \text{ and } m \in \mathbb{R}.$$

Thus, by the second step in the proof of Theorem 2.6 and Proposition C.6 holds

$$\tilde{\mathcal{A}} = \hat{\mathcal{A}} \quad \Longleftrightarrow \quad \tilde{\alpha}_{\min}^- = \left[\text{cl}\left(\tilde{\alpha}_{\min} \square \delta_{\mathcal{X}}\right)\right]^- + \delta_{\mathcal{K}} = \text{cl}_{\mathcal{P}^{\min}}\left(\tilde{\alpha}_{\min} \square \delta_{\mathcal{X}} + \delta_{\mathcal{K}}\right).$$

Since $\tilde{R}$ is uniquely determined as the left inverse of $\tilde{\alpha}_{\min}^-$, Proposition C.6 yields

$$\tilde{\mathcal{A}} = \hat{\mathcal{A}} \quad \Longleftrightarrow \quad \tilde{R} = \text{cl}_{\mathcal{R}^{\max}}\left(\left[(\tilde{\alpha}_{min} \square \delta_{\mathcal{X}}) + \delta_{\mathcal{K}}\right]^{-1,l}\right) \text{ on } \mathcal{K}^\circ \times \mathbb{R}.$$

We are left to show that

$$\left[(\tilde{\alpha}_{min} \square \delta_{\mathcal{X}}) + \delta_{\mathcal{K}}\right]^{-1,l}(x^*, s) = \sup_{y^* \in \mathcal{V}^*} R\left(x^* - y^*, s - \delta_{\mathcal{X}}(y^*)\right) \text{ for all } x^* \in \mathcal{K}^\circ \text{ and } s \in \mathbb{R}.$$

For any $x^* \in \mathcal{K}^\circ$ and $s \in \mathbb{R}$ holds

$$\left[(\tilde{\alpha}_{min} \square \delta_{\mathcal{X}}) + \delta_{\mathcal{K}}\right]^{-1,l}(x^*, s) = \inf\left\{m \in \mathbb{R} : s \le (\tilde{\alpha}_{min} \square \delta_{\mathcal{X}})(x^*, m) + \delta_{\mathcal{K}}(x^*)\right\}$$

$$= \inf\left\{m \in \mathbb{R} : s \le \inf_{y^* \in \mathcal{V}^*} \tilde{\alpha}_{\min}(x^* - y^*, m) + \delta_{\mathcal{X}}(y^*)\right\}$$

$$= \inf\left\{m \in \mathbb{R} : s \le \tilde{\alpha}_{\min}(x^* - y^*, m) + \delta_{\mathcal{X}}(y^*) \text{ for all } y^* \in \mathcal{V}^*\right\}$$

$$= \inf\left\{m \in \mathbb{R} : s \le \tilde{\alpha}_{\min}^+(x^* - y^*, m) + \delta_{\mathcal{X}}(y^*) \text{ for all } y^* \in \mathcal{V}^*\right\}.$$

Finally, since $\left[\tilde{\alpha}_{\min}^+(x^* - y^*, \cdot) + \delta_{\mathcal{X}}(y^*)\right]^{(-1,l)} = \tilde{R}(x^* - y^*, \cdot - \delta_{\mathcal{X}}(y^*))$, by use of relation (B.5) follows

$$\left[(\tilde{\alpha}_{min} \square \delta_{\mathcal{X}}) + \delta_{\mathcal{K}}\right]^{-1,l}(x^*, s) = \inf\left\{m \in \mathbb{R} : R(x^* - y^*, s - \delta_{\mathcal{X}}(y^*)) \le m \text{ for all } y^* \in \mathcal{V}^*\right\}$$

$$= \sup_{y^* \in \mathcal{V}^*} R(x^* - y^*, s - \delta_{\mathcal{X}}(y^*)). \qquad \square$$

C.14 Proof of Theorem 3.2: We first show that $\preccurlyeq$ is $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-lower semicontinuous. Indeed, the Fatou property and the dominated convergence theorem imply that $\mathcal{L}(Y) \cap B_N$ is $\|\cdot\|_1$-closed for all $Y \in \mathbb{L}^\infty$ and any ball $B_N = \{X : \|X\|_\infty \le N\}$ of radius $N > 0$. By convexity $\mathcal{L}(Y) \cap B_N$ is $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$-closed and consequently $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-closed. The Krein-Šmulian theorem then implies that $\mathcal{L}(Y)$ is $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-closed. Moreover, since $\mathbb{L}^1$ is separable by means of the separability of $\mathscr{F}$, it follows from Corollary 3.2 and its subsequent remark in [11], that $\preccurlyeq$ is separable. By Proposition 2.2 it can be represented by a $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-lower semicontinuous risk measure $\rho : \mathbb{L}^\infty \to [-\infty, +\infty]$. Finally, the robust representation (3.2) follows from Theorem 2.7.

C.15 Proof of Proposition 3.5: Let (K_n) be an increasing sequence of compact intervals such that $\bigcup_n K_n = \mathbb{R}$ and denote by $\preceq_n$, $\mathcal{M}_1(K_n)$, $ca(K_n)$, $C(K_n)$ the respective restrictions of $\preceq$, $\mathcal{M}_{1,c}$, ca_c, C to K_n. Due to Corollary 3.2 and its subsequent remark in Campion, Candeal, and Indurain [11], the risk order $\preceq_n$ is separable and consequently so is $\preceq$. By Proposition 2.2, the risk order $\preceq$ can be represented by a lower semicontinuous risk measure $\rho : \mathcal{M}_{1,c} \to [-\infty, +\infty]$. Denote by ρ_n the restriction of ρ to $\mathcal{M}_1(K_n)$ which is again a lower semicontinuous risk measure. Recall that $ca(K_n)$ is the dual space of $C(K_n)$ and $\mathcal{M}_1(K_n)$ is $\sigma(ca(K_n), C(K_n))$-compact in $ca(K_n)$. The $\sigma(ca(K_n), C(K_n))$-lower semicontinuous risk order $\preceq_n$ is therefore continuously extensible as well as ρ_n. Then, Theorem 2.19 yields a unique $R_n \in \mathcal{R}^{\max}_{\mathcal{M}_1(K_n)}$ such that

$$\rho_n(\mu) = \sup_{f \in \mathcal{K}^\circ(K_n)} R_n\left(f, -\int f(x)\mu(dx)\right) \quad \text{for all } \mu \in \mathcal{M}_1(K_n). \tag{C.19}$$

Since $\rho_{n'}$ coincides with ρ_n on $\mathcal{M}_1(K_n)$ for all $n' \geq n$, for the respective risk functions holds

$$R_{n'}(f, \cdot) \leq R_n(f, \cdot) \quad \text{for all } f \in \mathcal{K}^\circ,$$

where $R_n(f, \cdot) := R_n(f_{|K_n}, \cdot)$ for all $f \in \mathcal{K}^\circ$, since $\alpha^{n'}_{\min}(f, \cdot) \geq \alpha^n_{\min}(f, \cdot)$. We next show that ρ and $\preceq$ are continuously extensible. To this end, we define $\tilde{R}(f, \cdot)$ as the left-continuous version of $\inf_{n \in \mathbb{N}} R_n(f, \cdot)$. For $f \in \mathcal{K}^\circ(K_n)$ denote by $\bar{f} \in \mathcal{K}^\circ$ the extension f which is constant outside of K_n. For any $\mu \in \mathcal{M}_{1,c}$ holds by partial integration

$$\int \bar{f}(x)\mu(dx) = \sup_{x \in K_n} f(x) - \int F_\mu(x)d\bar{f}(x) = \sup_{x \in K_n} f(x) - \int_{K_n} F_\mu(x)df(x) = \int_{K_n} f(x)\mu(dx),$$

showing that $\tilde{R}(\bar{f}, \cdot) = R_n(f, \cdot)$. Fix $\mu \in \mathcal{M}_{1,c}$ and $n \in \mathbb{N}$ such that $\mu \in \mathcal{M}_{1,c}(K_n)$. By (C.19) holds

$$\rho(\mu) = \rho_n(\mu) = \sup_{f \in \mathcal{K}^\circ(K_n)} R_n\left(f, -\int f(x)\mu(dx)\right) = \sup_{f \in \mathcal{K}^\circ} \tilde{R}\left(f, -\int f(x)\mu(dx)\right),$$

for any $\mu \in \mathcal{M}_{1,c}$. Due to the previous representation, it follows that ρ and in turn $\preceq$ are $\sigma(ca_c, C)$-lower semicontinuous and continuously extensible to ca_c. In view of Theorem 2.19 there exists a unique $R \in \mathcal{R}^{\max}_{\mathcal{M}_{1,c}}$ such that

$$\rho(\mu) = \sup_{f \in \mathcal{K}^\circ} R\left(f, -\int f(x)\mu(dx)\right) = \sup_{l \in \mathcal{K}^\circ} R\left(l, \int l(-x)\mu(dx)\right) \quad \text{for all } \mu \in \mathcal{M}_{1,c},$$

since $f \in \mathcal{K}^\circ$ if and only if $l(x) = -f(-x) \in \mathcal{K}^\circ$.

It remains to prove that $\mathcal{R}^{\max}_{\mathcal{M}_{1,c}} = \mathcal{R}^{\max}$. Indeed, for $f, g \in C$ with $f \leq g$ and $c, m, s \in \mathbb{R}$ holds $\alpha_{\min}(f, m) \geq \alpha_{\min}(g, m)$ and $\alpha_{\min}(f + c, m) = \alpha_{\min}(f, m) - c$ and in turn $R(f, s) \leq R(g, s)$ and $R(f + c, s) = R(f, s + c)$. Moreover, $\delta_{\mathcal{M}_{1,c}}(g) = \sup_{\mu \in \mathcal{M}_{1,c}} -\int g(x)\mu(dx) = -\inf g$. Hence,

$$\sup_{g \in C} R\left(f - g, s - \delta_{\mathcal{M}_{1,c}}(g)\right) = \sup_{g \in C} R\left(f - g, s + \inf_{x \in I} g(x)\right) = \sup_{c \in \mathbb{R}} R\left(f - c, s + c\right) = R(f, m).$$

The claim then follows from Definition 2.17 and the proof is completed.

C.16 Proof of Robust Representation in Example 3.6: According to [2, Corollary 15.6], the set

$$\mathcal{A}^m = \{\mu \in \mathcal{M}_{1,c} : q \geq \mu\,(]-\infty, -m[)\},$$

is $\sigma(ca_c, C)$-closed in $\mathcal{M}_{1,c}$, implying that $V@R$ is a lower semicontinuous risk measure on $\mathcal{M}_{1,c}$. Due to Theorem 3.5, it admits then a robust representation. To compute the penalty function

$$\alpha_{\min}(l, m) = \sup_{\mu \in \mathcal{A}^m} \int l(-x)\,\mu(dx),$$

we define $\mu_t := q\delta_t + (1-q)\delta_{-m}$ which is in $\mathcal{A}^m$ since $F_{\mu_t}^-(-m) \le q$ for all $t \in \mathbb{R}$. Then for any loss function $l \in \mathcal{K}^{1,\circ}$

$$\alpha_{\min}(l,m) = \sup_{\mu \in \mathcal{A}^m} \int l(-x)\, d\mu = \lim_{t \to -\infty} \int l(-x)\, \mu_t(dx) = ql(+\infty) + (1-q)l(m).$$

Thus, for any $l \in \mathcal{K}^{\circ}$ holds

$$R(l,s) = l^{-1}\left(\frac{s - ql(+\infty)}{1-q}\right),$$

where l^{-1} is the left inverse of l.

C.17 Proof of the Theorem 3.7: Since $\mathcal{K} = \mathcal{CS}$ and $\preccurlyeq$ is monotone and lower semicontinuous holds $\overline{\mathcal{L}(c) + \mathcal{K}} = \overline{\mathcal{L}(c)} = \mathcal{L}(c)$ for all $c \in \mathcal{CS}$. Therefore, $\preccurlyeq$ is continuously extensible to $\mathcal{V}_\eta$ and we can apply Theorem 2.19.

C.18 Proof of Proposition 3.8: From the assumptions on l, inspection shows that ρ is a risk measure. Consider now a sequence (c^n) in $\mathcal{CS}$ converging to $c \in \mathcal{CS}$ and define $y^n = \int_{-k_1(\cdot)}^{\cdot + k_2(\cdot)} \theta(\cdot,s)\, dc_s^n$. From the assumptions on θ, it follows

$$|y_t^n| \le \int\limits_{0-}^{1} |\theta(t,s)|\, dc_s^n \le Cc_1^n \le M \sup_{n \in \mathbb{N}} \|c^n\|_\eta < +\infty,$$

for some constants $C, M > 0$. Hence (y^n) is uniformly bounded and converges pointwise to $y = \int_{-k_1(\cdot)}^{\cdot + k_2(\cdot)} \theta(\cdot,s)\, dc_s$. Since l is lower semicontinuous, $\liminf_n l(t, -y_t^n) \ge l(t, -y_t)$. Furthermore, since l is continuous in the first argument and nondecreasing in the second it follows $l(t, y_t^n) \ge l(t, -\sup y^n) \ge M \in \mathbb{R}$. Applying Fatou's lemma yields

$$\liminf_n \int\limits_0^1 l(t, -y_t^n)\, dt \ge \int\limits_0^1 \liminf_n l(t, -y_t^n)\, dt \ge \int\limits_0^1 l(t, -y_t)\, dt$$

and so, ρ is lower semicontinuous.

C.19 Computations for Example 3.9: In order to simplify the computation of $\alpha_{\min}$, we first relax the risk measure from $\mathcal{CS}$ to $\mathcal{V}_\eta$. Integration by parts yields

$$\tilde{\rho}(c) := \int\limits_0^1 l\left(-c_t + \gamma \int\limits_0^t e^{-\gamma(t-s)} c_s ds\right) dt = \int\limits_0^1 l(-y_t)\, dt, \quad c \in \mathcal{V}_\eta, \tag{C.20}$$

where $y_t = c_t - \gamma \int_0^t e^{-\gamma(t-s)} c_s ds$. In the line with the proof of Proposition 3.8 shows that $\tilde{\rho}$ is a lower semicontinuous risk measure on $\mathcal{V}_\eta$ which is an extension of ρ that is however not maximal. Further, for any $c \in \mathcal{V}_\eta$ the respective y given by

$$y_t = c_t - \gamma \int\limits_0^t e^{-\gamma(t-s)} c_s ds \tag{C.21}$$

is in $\mathcal{V}_\eta$. Conversely, for any $y \in \mathcal{V}_\eta$ the Volterra equation (C.21) of the second kind has the unique solution $c_t = y_t + \gamma \int_0^t y_s ds$, which is in $\mathcal{V}_\eta$. By use of this one-to-one relation, $c \in \tilde{\mathcal{A}}^m$ exactly when $\int_0^1 l(-y_t)dt \le m$ and using (3.11) the minimal penalty function $\tilde{\alpha}_{\min}$ for $\beta \in \mathcal{D}$ can be computed as

$$\tilde{\alpha}_{\min}(\beta, m) = \sup_{c \in \tilde{\mathcal{A}}^m} -\int\limits_{0-}^1 \beta_t dc_t = \sup_{\{y \in \mathcal{V}_\eta: \int_0^1 l(-y_t)dt \le m\}} -\beta_1 y_1 - \int\limits_0^1 y_t \Delta\beta_t dt, \tag{C.22}$$

where $\Delta\beta_t := \gamma\beta_t - \beta_t'$, since $dc_t = dy_t + \gamma y_t dt$. Fix $m > \inf l$, since otherwise, for $m \le \inf l$ holds $\tilde{\alpha}_{\min}(\beta, m) = -\infty$ as $\mathcal{A}^m = \emptyset$.

- If $\beta_1 > 0$ then $\tilde{\alpha}_{\min}(\beta, m) = +\infty$. Indeed, take some $c \in \tilde{\mathcal{A}}^m$. By (C.20) we have $c_k = c + k\delta_1 \in \tilde{\mathcal{A}}^m$ for

all $k \in \mathbb{R}$, so that

$$\tilde{\alpha}_{\min}(\beta, m) \geq -k\beta_1 + \int_0^1 c_s \beta_s' ds \xrightarrow[k \to -\infty]{} +\infty.$$

- If $\Delta\beta = \gamma\beta - \beta' < 0$ over a set of positive measure then $\tilde{\alpha}_{\min}(\beta, m) = +\infty$. Indeed, there is $\varepsilon > 0$ such that $A := \{\Delta\beta \leq -\varepsilon\}$ has positive measure and define $y_t^M = -M/(\Delta\beta_t) 1_A(t)$ for $M \geq 0$. It follows $\tilde{\rho}(c^M) = \int_0^1 l(-y_t) dt \leq m$ and in turn $c^M \in \tilde{\mathcal{A}}^m$ for all M sufficiently large, showing that

$$\tilde{\alpha}_{\min}(\beta, m) \geq -\int_0^1 y_t \Delta\beta_t dt = M \int_0^1 1_A(t) dt \xrightarrow[M \to +\infty]{} +\infty.$$

We suppose now that β is such that $\Delta\beta \geq 0$, $\beta_1 = 0$ and $m > 0$. For some Lagrange multiplier $\delta := \delta(\beta, m) > 0$, define the function

$$\chi(y) := \chi(y, \beta, \delta, m) = \int_0^1 \left[-y_t \Delta\beta_t - \frac{1}{\delta}(l(-y_t) - m) \right] dt,$$

for which clearly holds $\tilde{\alpha}_{\min}(\beta, m) \leq \sup_{\{y : \int_0^1 l(-y_t) dt \leq m\}} \chi(y)$. The first order condition yields

$$l'(-\hat{y}_t) = \delta\Delta\beta_t \iff \hat{y}_t = -(l')^{-1}(\delta\Delta\beta_t).$$

As for δ, under positivity and integrability conditions, it is determined by the equation $\int_0^1 l\left((l')^{-1}(\delta\Delta\beta_t)\right) dt = m$. For such a choice of δ, holds $c \in \tilde{\mathcal{A}}^m$ and therefore

$$\chi(\hat{y}) = \int_0^1 -\hat{y}_t \Delta\beta_t dt \geq \sup_{\{y : \int_0^1 l(-y_t) dt \leq m\}} \chi(y) \geq \tilde{\alpha}_{\min}(\beta, m) \geq \int_0^1 -\hat{y}_t \Delta\beta_t dt.$$

In the case where

- $l(x) = e^x$ holds $(l')^{-1}(x) = \ln(x)$, $\delta = m/\int_0^1 \Delta\beta_t dt$ and

$$\tilde{\alpha}_{\min}(\beta, m) = \ln(\delta) \int_0^1 \Delta\beta_t dt + \int_0^1 \ln(\Delta\beta_t) \Delta\beta_t dt = \ln(m) \int_0^1 \Delta\beta_t dt + g(\beta),$$

where

$$g(\beta) := \int_0^1 \ln(\Delta\beta_t) \Delta\beta_t dt - \ln\left(\int_0^1 \Delta\beta_t dt\right) \int_0^1 \Delta\beta_t dt,$$

and thus

$$\tilde{R}(\beta, s) = \exp\left(\frac{s - g(\beta)}{\int_0^1 \Delta\beta_t dt}\right).$$

- $l(x) = -\ln(-x)$ holds $(l')^{-1}(x) = -1/x$, $\delta = \exp\left(m - \int_0^1 \ln(\Delta\beta_t) dt\right)$ and

$$\tilde{\alpha}_{\min}(\beta, m) = -\int_0^1 \hat{y}_t \Delta\beta_t dt = -\frac{1}{\delta} = -\frac{\exp\left(\int_0^1 \ln(\Delta\beta_t) dt\right)}{e^m},$$

and thus

$$\tilde{R}(\beta, s) = -\ln(-s) + \int_0^1 \ln(\Delta\beta_t) dt.$$

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C.20 Proof of Theorem 3.10: The restriction of $\preccurlyeq$ to $\mathcal{M}_{1,c}$ is $\sigma\left(\mathcal{M}_{1,c}, C\right)$-lower semicontinuous and monotone with respect to the first stochastic order. In view of Theorem 3.5 the restriction of $\preccurlyeq$ to $\mathcal{M}_{1,c}$ is separable and can be represented by a lower semicontinuous risk measure $g : \mathcal{M}_{1,c} \to \mathbb{R}$. The function $h\left(c\right) = g\left(\delta_c\right)$ for $c \in \mathbb{R}$ is decreasing and lower semicontinuous. Due to (ii) and (iii) holds $Im\left(g\right) = Im\left(h\right)$. Indeed, take some $\mu \in \mathcal{M}_{1,c}$ and suppose that $g\left(\mu\right) \notin Im\left(h\right)$. Consider the smallest $t \in \mathbb{R}$ such that $\delta_t \preccurlyeq \mu$ which exists due to the lower semicontinuity. From the sensitivity, $\mu \sim \delta_t$ in contradiction to $g\left(\mu\right) \notin Im\left(h\right)$. This shows that up to an increasing lower semicontinuous transformation, we can suppose that $Im(g) = \mathbb{R}$ and $g\left(\delta_c\right) = -c$. The lower semicontinuity implies moreover that $\omega \mapsto G(\tilde{\mu})\left(\omega\right) := g\left(\tilde{\mu}\left(\omega\right)\right)$ is measurable. The condition (3.17) and the monotonicity imply that for any $k > 0$ such that the support of $\tilde{\mu}$ lies uniformly in $[-k, k]$ holds $k = g\left(\delta_{-k}\right) \geq G(\tilde{\mu}) \geq g\left(\delta_k\right) = -k$, showing that G maps $\mathcal{SK}$ to $\mathbb{L}^\infty$. Moreover, the fact that[58] $G(\delta_X) = -X$ for any $X \in \mathbb{L}^\infty$ yields $Im\left(G\right) = \mathbb{L}^\infty$.

We now define the binary relation $\preccurlyeq^G$ on $\mathbb{L}^\infty$ by

$$X \preccurlyeq^G Y \quad \Longleftrightarrow \quad \tilde{\mu} \preccurlyeq \tilde{\nu} \quad \text{for } \tilde{\mu} \in G^{-1}\left(-X\right) \text{ and } \tilde{\nu} \in G^{-1}\left(-Y\right).$$

In order to be well defined, we have to show that for any $X \in \mathbb{L}^\infty$, the stochastic kernels in $G^{-1}\left(-X\right)$ are equivalent to each other. To do so, consider two stochastic kernels $\tilde{\mu}, \tilde{\nu} \in \mathcal{SK}$ such that $G(\tilde{\mu}) = G(\tilde{\nu})$ P-almost surely. Then $g\left(\tilde{\mu}\left(\omega\right)\right) = g\left(\tilde{\nu}\left(\omega\right)\right)$ for P-almost all $\omega \in \Omega$ and so $\tilde{\mu}\left(\omega\right) \sim \tilde{\nu}\left(\omega\right)$. In view of condition (3.17), it follows that $\tilde{\mu} \sim \tilde{\nu}$ and therefore, for any $X \in \mathbb{L}^\infty$, the elements of $G^{-1}\left(-X\right)$ are equivalent. Moreover, $\preccurlyeq$ is transitive and complete, and therefore a total preorder. It is furthermore a risk order. Indeed, concerning the monotonicity, for $X, Y \in \mathbb{L}^\infty$ with $X = -G(\tilde{\mu}) \geq -G(\tilde{\nu}) = Y$, the condition (3.17) implies $\tilde{\mu} \preccurlyeq \tilde{\nu}$ and therefore $X \preccurlyeq^G Y$. As for the quasi-convexity, consider some $X, Y \in \mathbb{L}^\infty$ with $X \preccurlyeq^G Y$ and set $\tilde{\mu} := \delta_X$, $\tilde{\nu} := \delta_Y$ such that $\tilde{\mu} \in G^{-1}\left(-X\right)$ and $\tilde{\nu} \in G^{-1}\left(-Y\right)$, showing that $\tilde{\mu} \preccurlyeq \tilde{\nu}$. The P-almost sure second stochastic order yields $\delta_{\lambda X+(1-\lambda)Y} \trianglerighteq \lambda\delta_X + (1-\lambda)\delta_Y = \lambda\tilde{\mu} + (1-\lambda)\tilde{\nu}$, since P-almost surely holds

$$\int f\, d\delta_{\lambda X+(1-\lambda)Y} = f\left(\lambda X + (1-\lambda)Y\right) \geq \lambda f\left(X\right) + (1-\lambda) f\left(Y\right)$$

$$= \lambda \int f\, d\delta_X + (1-\lambda) \int f\, d\delta_Y = \lambda \int f\, d\tilde{\mu} + (1-\lambda) \int f\, d\tilde{\nu},$$

for any nondecreasing concave function f. Since $\preccurlyeq$ is a risk order, it follows that $\delta_{\lambda X+(1-\lambda)Y} \preccurlyeq \lambda\tilde{\mu}+(1-\lambda)\tilde{\nu} \preccurlyeq \tilde{\nu}$. Hence $\lambda X + (1-\lambda)Y \preccurlyeq^G Y$. Finally, the risk order $\preccurlyeq^G$ satisfies the Fatou property. Indeed, let X_n be a $\|\cdot\|_\infty$-bounded sequence in $\mathbb{L}^\infty$ converging P-almost surely to some $X \in \mathbb{L}^\infty$. Due to condition (i) holds that $\delta_{X_n} \preccurlyeq \delta_Y$ for all n implies $\delta_X \preccurlyeq \delta_Y$. From the definition of $\preccurlyeq^G$ follows that $X_n \preccurlyeq^G Y$ for all n implies $X \preccurlyeq^G Y$, and therefore, by means of Theorem 3.2, the risk order $\preccurlyeq^G$ is a separable and $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-lower semicontinuous. Let $\Phi : \mathbb{L}^\infty :\to [-\infty, +\infty]$ be a $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$-lower semicontinuous risk measure representing $\preccurlyeq^G$. Then

$$\rho\left(\tilde{\mu}\right) := \Phi\Big(\omega \mapsto -g\big(\tilde{\mu}\left(\omega\right)\big)\Big), \quad \tilde{\mu} \in \mathcal{SK},$$

is a risk measure corresponding to $\preccurlyeq$. Indeed, for $\tilde{\mu} \in \mathcal{SK}$ holds $\tilde{\mu} \sim \delta_{-G(\tilde{\mu})}$, and therefore $\tilde{\mu} \preccurlyeq \tilde{\nu}$ is equivalent to $\delta_{-G(\tilde{\mu})} \preccurlyeq \delta_{-G(\tilde{\nu})}$ which by definition is equivalent to $-G\left(\tilde{\mu}\right) \preccurlyeq^G -G\left(\tilde{\nu}\right)$. Hence $\tilde{\mu} \preccurlyeq \tilde{\nu}$ is equivalent to $\Phi\left(-G\left(\tilde{\mu}\right)\right) \leq \Phi\left(-G\left(\tilde{\nu}\right)\right)$.

Conversely, it is plain to check that any risk order corresponding to a risk measure of the form (3.19) fulfills the conditions (*i*) to (*iv*).

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[58] Recall that for $X \in \mathbb{L}^\infty$, the notation δ_X stands for the stochastic kernel $\delta_{X(\omega)}\left(dx\right)$ which is a Dirac measure at $X\left(\omega\right)$ for all $\omega \in \Omega$.

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